PAPER No. 94.

ANALYSIS OF PARTLY STIFFENED SUSPENSION BRIDGES TYPE 2 F.

By J. Halcro Johnston, B. Sc.

At the Punjab Engineering Congress 1922 Mr. Lyster discussing his paper on Suspension Bridges gave as his opinion that such bridges must either be stiffened in the orthodox way or left unstiffened altogether. We disagreed with this view at the time and showed how most of the bridges in the Punjab Hills depended for stiffness on the rigidity of their floors. This paper attempts to give a solution of the general case where the Moment of Inertia of the stiffening system may vary from zero to infinity.

The method used is not new. It is dealt with in Burr's Suspension Bridges, pages 212—247, in Johnston Bryan and Turneaure: Modern Framed Structures, Part 2, pages 276—318 and in Melan-Steinman: Theory of Arches and Suspension Bridges, pages 76—86. The first two books only have been consulted. In both, the method is used to get more accurate results for a fully stiffened bridge. Influence lines are not worked out. As the author of the chapter referred to in Modern Framed Structures rightly says, the bending moment is not proportional to the load and the use of influence lines is therefore inadmissible. In the following theory, however, it will be shown that this is hardly a practical objection. The use of influence lines is becoming more and more the practice of every drawing office and this paper confines itself to methods of drawing these. The special case of small Moment of Inertia or what is sometimes referred to as the stiffened floor is also dealt with. No attempt is made to treat the general case of continuous girder and suspended side spans. The analysis is of a bridge with free ends at the towers and straight back stays, referred to by Steinman as type 2 F. This we think is the type which will be most used in the Hills of India and the treatment of one case only makes the analysis easier to follow.

The method and assumptions are briefly as follows :—moment of inertia uniform for all parts of the span, no bending moment and no end reactions under dead load and at mean temperature. The thrust is found in the ordinary way with the assumption of deflection negligible or large moment of inertia. No attempt is made to correct this for deflection effects. The bending moment is then worked out for a single load as a function of the thrust and deflection. The well-known equation connecting moment and deflection is added and the deflection is eliminated leaving an equation giving the moment as a function of the thrust, the position of the load and the position of the section at which the moment is required.

The following diagram and definitions will explain the use of the letters :—

au——————deflection at P.

$\frac{du}{dx}=i$——————slope „ „

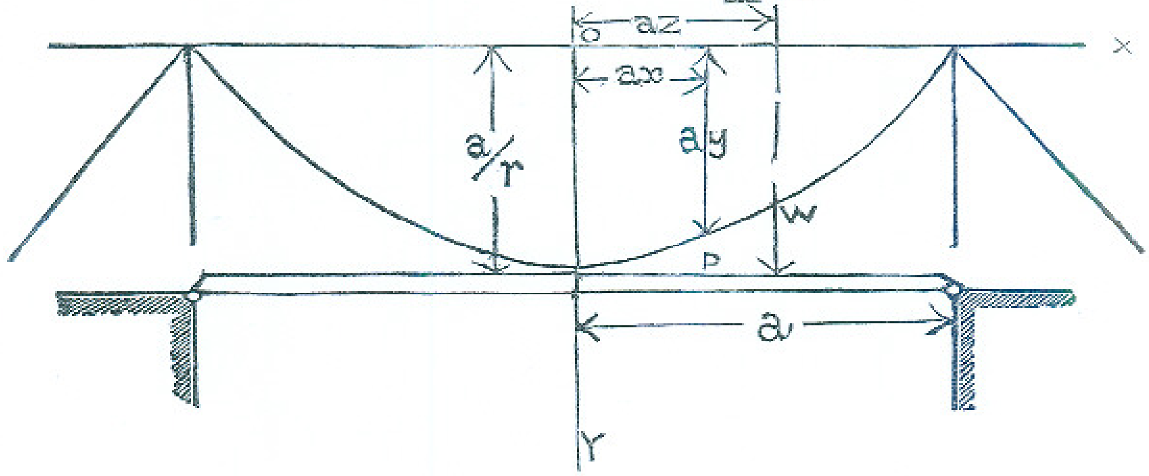
$\frac{EI}{a}\frac{d^2u}{dx^2}=Wam=M$——————BM „ „ due to W

$\frac{EI}{a^2}\frac{d^3u}{dx^3}=W\frac{dm}{dx}=Ws$——————shear „ „ „

H——————thrust.

Wrh——————thrust due to W

I——————moment of inertia of floor, etc.



Thrust.—The thrust is found by the method of least work. It depends on the work done in bending the stiffening girders, stretching the cables including backstays and that due to temperature strains. It must therefore be worked out independently for each case. The following solution will however generally give results within 5 per cent. and will illustrate the method of calculation. It allows for the work done in bending only.

Work done in bending the girder $=F=\frac{1}{2EI}\int M^2dx$ ....1.

Diff. with respect to H : $\frac{dF}{dH}=\frac{1}{EI}\int M\frac{dM}{dH}dx=0.$

Let M°=bending moment in girder assuming no cable then :

$$M=M^\circ-Hay \text{ ....2.}$$

therefore $\frac{dM}{dH}=-ay$

$$\int(M^\circ-Hay)\,ay\,dx=0$$

$$H=\frac{\int M^\circ y\,dx}{a\int y^2\,dx} \text{ ....3.}$$

To find the influence line for H assume a single load W.

Let $M^\circ = Wam^\circ$ where $m^\circ = \frac{1}{2}(1-z)(1+x)$ ...... x less than z

$$= \frac{1}{2}(1+z)(1-x) \ldots\ldots \left\{ \begin{matrix} \\ z \quad ,, \quad ,, \quad x \end{matrix} \right.$$
$$= \frac{1}{2}(1-z)(1+x)-(x-z)$$

$$y = (1-x^2)/r$$

$$\text{hence } h = \frac{H}{Wr} = \frac{\int m^\circ (1-x^2)\,dx}{\int (1-x^2)^2\,dx} \ldots\ldots\ldots\ldots 4.$$

The values of these integrals are easily found :—

$$\int_{-1}^{+1} m^\circ (1-x^2)\,dx = \int_{-1}^{+1} \tfrac{1}{2}(1-z)(1+x)(1-x^2)\,dx - \int_{z}^{+1} (x-z)(1-x^2)\,dx$$

$$= (1-z)\int_{0}^{1}(1-x^2)\,dx - \int_{z}^{1}(x-z-x^3+zx^2)\,dx$$

$$= \underline{(5-z^2)(1-z^2)/12}$$

$$\int_{-1}^{+1}(1-x^2)^2\,dx = 2\int_{0}^{1}(1-2x^2+x^4)\,dx = \underline{16/15}$$

hence $\underline{h = (5-z^2)(1-z^2) \times 5/64}$ ....5.

This is the influence line of H. It is shown on plate I plotted for different values of z.

In the case of a uniform load "W" over the whole span :

$$H = \int_{-1}^{+1} rh\,dW = 2wra\int_{0}^{1} h\,dz = \tfrac{5}{32}wra\int_{0}^{1}(5-z^2)(1-z^2)\,dz$$

$$= \underline{\tfrac{1}{2}wra} \ldots\ldots\ldots\ldots\ldots\ldots 6.$$

This the same as $wl^2/8f$, the formula generally used in approximate calculations of thrust.

Bending Moment.—The Bending moment due to a single live load W given above is more accurately expressed as :—

$$M = M^\circ - H^1 ay - Hau \ldots\ldots\ldots\ldots 7.$$

where "au" is the deflection at P, H the total thrust due to all causes dead and live. H^1 is that due only to the load W. The terms due to dead load are left out as they cancel each other. The equation may be rewritten :—

$$Wam = Wam^{\circ} - rhayW - Hau$$

or $$m = m^{\circ} - rhy - Hu/W \ldots\ldots 8.$$

But $M = -EI\dfrac{d^2Y}{dX^2}$ where $X = ax$ and $Y = au$

therefore $\dfrac{d^2u}{dx^2} = -\dfrac{a^2W}{EI}m$

substitute (8) $\dfrac{d^2u}{dx^2} = \dfrac{a^2W}{EI}(rhy - m^{\circ}) + \dfrac{a^2H}{EI}u$; put $c^2 = \dfrac{a^2H}{EI}$

then $$\frac{d^2u}{dx^2} - c^2u = c^2F(x) \ldots\ldots 10.$$

where $F(x) = (rhy - m^{\circ})W/H$.

Since F (x) is of the second degree it can easily be shown by differentiating that the following is a solution of equation 10 :

$$u = P\,e^{cx} + Q\,e^{-cx} - F(x) - \frac{1}{c^2}F''(x) \ldots\ldots 11.$$

where P and Q are constants of integration.

Now (8) may be rewritten $-m = [F(x) + u]\,H/W \ldots\ldots$ 12.

substitute (11) and note that $F''(x) = -2hW/H$

$$-m = (P'\,e^{cx} + Q'\,e^{-cx})H/W + 2h/c^2$$
$$= A \sinh cx + B \cosh cx + D \ldots\ldots 13.$$

A and B are constants of integration and $D = 2h/c^2$. Now since "u" and F (x) consist of two separate branches according as "x" or "z" is greater, so also will "u" and "m." We have therefore four constants to eliminate.

Constants of integration.—To eliminate these constants four equations are required. These are got by equating the Bending Moments to zero when $x = +1$ and -1 and when $x = z$ by equating the BMs of the two equations and by putting the difference of the shears equal to W.

Let $-m = A \sinh cx + B \cosh cx + D$ "x" less than "z" 14.

$-m' = A' \sinh cx + B' \cosh cx + D$ "z" ,, ,, "x" 15.

The shear $= -\dfrac{dM}{dX} = -W\dfrac{dm}{dx}$ and the four conditions may be written :-

$$-A \sinh c + B \cosh c + D = 0 \ldots\ldots 16.$$

$$A' \sinh c + B' \cosh c + D = 0 \ldots\ldots 17.$$

$$A \sinh cz + B \cosh cz = A' \sinh cz + B' \cosh cz \ldots\ldots 18.$$

$$A \cosh cz + B \sinh cz = A' \cosh cz + B' \sinh cz - 1/c \ldots 19.$$

whence the following values of the constants are obtained :

$$A = -\frac{\sinh c\,(1-z)}{2c \sinh c} \qquad A' = \frac{\sinh c\,(1+z)}{2c \sinh c}$$

$$B = -\frac{\sinh c\,(1-z)}{2c \cosh c} - \frac{D}{\cosh c} \quad B' = -\frac{\sinh c\,(1+z)}{2c \cosh c} - \frac{D}{\cosh c}$$

Substitute these values in 14 and 15 also the value of D

$$m = \frac{\sinh c\,(1-z)}{2c}\left[\frac{\cosh cx}{\cosh c} + \frac{\sinh cx}{\sinh c}\right] - \frac{2h}{c^2}\left[1 - \frac{\cosh cx}{\cosh c}\right] \ldots .20.$$

$$m' = \frac{\sinh c\,(1+z)}{2c}\left[\frac{\cosh cx}{\cosh c} - \frac{\sinh cx}{\sinh c}\right] - \frac{2h}{c^2}\left[1 - \frac{\cosh cx}{\cosh c}\right] \ldots .21.$$

These are the most general equations of the Bending Moment influence lines. They show that it is independent of the dip ratio " r ". The effect of the moment of inertia is contained in the factor " c ". When c is O, I is infinite and the equations become those ordinarily used for stiffened bridges. When c is large, I is small and the equations are those of the unstiffened bridge. These cases will now be dealt with separately.

I. c=o Putting sinh c=c and cosh $c=1+c^2/2$, we get :

$$m = \tfrac{1}{2}(1-z)(1+x) - h(1-x^2) \quad \text{i.e.} \quad m^\circ - rhy \ldots\ldots 23.$$

$$m' = \tfrac{1}{2}(1+z)(1-x) - h(1-x^2) \quad \text{i.e.} \quad m^\circ - rhy \ldots\ldots 24.$$

These are the equations of the influence lines of the fully stiffened bridge and may be got from first principles. They are shown on plate 1 plotted for five different values of x.

We find on integrating that $\int_{-1}^{+1} m^\circ dz. = \int_{-1}^{+1} rhy\, dz = \tfrac{1}{2}(1-x^2)$.

Therefore $\int_{-1}^{+1} m\, dz = 0$ which means that a uniform live load over the whole span causes no bending at any point. This of course would not have been the case if the strain in the cable had been taken into account in finding the thrust. An inspection of the influence lines shows that the maximum positive moment occurs when x=z and when the load is about one-fifth from either end of the span. The maximum positive value of $\int mdz$ is approximately 0·068. This means that the greatest positive moment occurs when the bridge is loaded from A to B with a uniform load p lbs/ft. and is equal to 0·068 $pa^2 = pl^2/59$.

Shear.—On differentiating with respect to x we obtain the influence lines for shear. They are :—

$$s = 2hx + \tfrac{1}{2}(1-z) \quad x < z \ldots\ldots\ldots\ldots 25.$$

$$s' = 2hx - \tfrac{1}{2}(1+z) \quad x > z \ldots\ldots\ldots\ldots 26.$$

When x=1, $s^1 = 2h - \tfrac{1}{2}(1+z)$. This is the influence line of right reaction on the pier. Again on integrating we find that $\int_{-1}^{+1} s\, dz = 0$ which shows that a uniform live load over the whole span casues no shear at any section and no end reactions, and again this is due to the cable strains having been neglected.

II. c large.—Putting sinh c=cosh c=$\frac{1}{2}$ e^c etc., we obtain the following approximate equations of the B. M. influence lines :—

$$m=\frac{1}{2c}\left[e^{-c(z-x)}-\frac{4h}{c}\right] \qquad x<z\ldots\ldots 27.$$

$$m'=\frac{1}{2c}\left[e^{-c(x-z)}-\frac{4h}{c}\right] \qquad x>z\ldots\ldots 28.$$

These equations hold when the moment of inertia is very small as in the case where a stiffened roadway is used. They may be used when c is greater than 6. With c= infinity the bending moment vanishes. Plate 2 shows influence lines for different values of x when c=6,

$\int_{-1}^{+1} m\,dz=0$ as before showing that a uniform load causes no bending stress. The greatest positive moment occurs when z=x. Then

$$m = \frac{1}{2c}\left[1-\frac{4h}{c}\right]$$

which is nearly constant for all values of z but somewhat greater near the ends of the bridge. This is the B. M. due to a single concentrated load. It also appears that the moment varies inversely as c, that is it is proportional to $\sqrt{I}$. But for fixed stresses m is proportional to I/y′ where y′ is the distance of extreme fibre from the neutral axis. Hence y′ is proportional to $\sqrt{I}$.

Shear.—The shear influence lines are got by differentiating with respect to x :—

$$s = -\tfrac{1}{2}e^{-c(z-x)} \qquad x<z\ldots\ldots 29.$$

$$s' = \tfrac{1}{2}e^{-c(x-z)} \qquad x>z\ldots\ldots 30.$$

These equations show that for a load to cause any shear it must be very close to the section and that there are no end reactions unless the load is very near the end of the bridge. Also the maximum shear got by putting x=z is W/2.

Practical Application.—Hitherto suspension bridges have been divided into two distinct classes, stiffened and unstiffened. The latter have been used wherever cheapness has been the primary consideration and the former where stiffness has been essential. By the use of the preceding formulæ it becomes possible to design for any specified stiffness or depth of girder, and to determine the economic degree of stiffness to adopt we must have a much clearer conception of the relative advantages and disadvantages of this property. The principal considerations will probably be the cost of stiffening girders when they are used and the excessive gradients and dangerous oscillations without them. The author is not in a position to discuss these and the problem requires more than a mathematical treatment. A few deductions may however readily be made from the form of the equations. On plate 3 values of $l^\circ d^\circ$, l°, m_2, and m_1, functions of l/c have been plotted. These are defined as follows :—

$l' = al^\circ =$ .. length of span covered for maximum postive B. M.

$d = \frac{fra}{Eq} d^\circ$ ..depth of girder required for maximum positive B. M.

$I = \frac{rwa^3}{E} I^\circ$ ..Moment of inertia required for maximum positive B. M.

$M = pa^2 m_2$ ..maximum BM due to uniform load

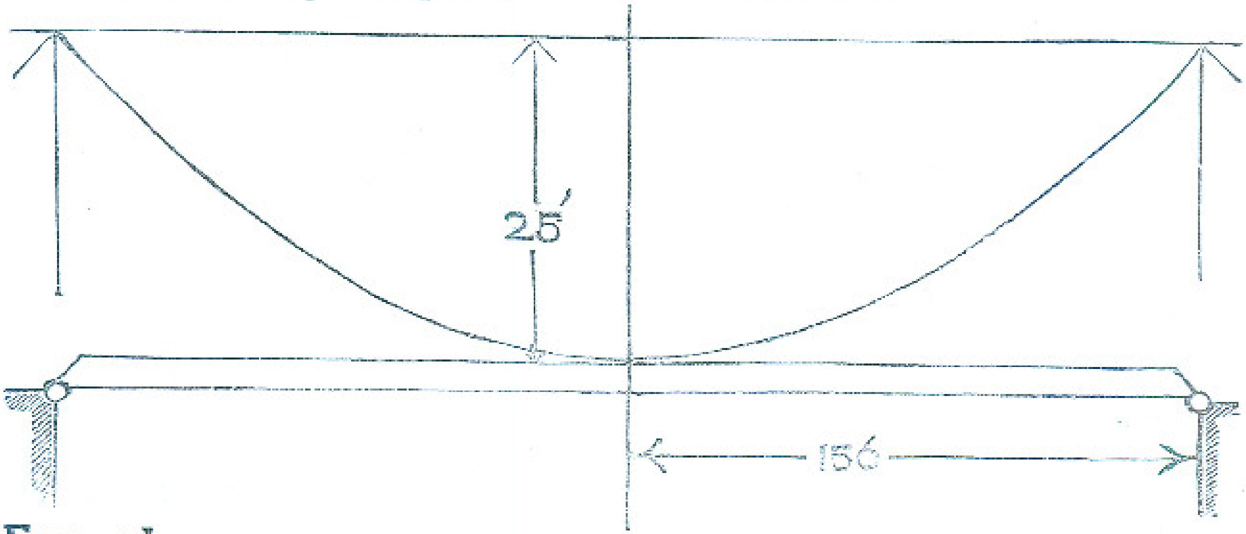
$= Wa\, m_1$ ..maximum BM due to single load W

$q = p/w$ ..ratio of live to dead load.

An inspection of the moment curves shows that for values of l/c between 0 and 0·1 of the concentrated load curve and between 0·1 and 0·6 of the uniform load curve the moment is approximately proportional to l/c. Now assume that we use beams of the same shape, *i. e.,* radius of gyration proportional to depth. Then from first principles of bending "M" varies as Ad where "A" is the area of cross section and "d" the depth. Also M varies as l/c which is proportional to $d\sqrt{A}$, hence A is independent of the depth. In an ordinary girder bridge A is proportional to l/d and there is economy in using a deep girder. In a suspension bridge, if our assumptions hold, the material remains nearly constant whatever the depth.

N. B.—The curves in plate 3 are to be used as a guide only and not to replace calculation.

The following example will make the method clear :—



Example.

$a = 150$ feet

$r = 6$

$p = 1000$ lbs./ft. live load

$w = 600$ lbs./ft. dead load

$W = 3000$ lbs. live load

$E = 30 \times 10^6$ lb./in^2 for steel

$= 2 \times 10^6$ lb./in^2 for timber

$f = 18 \times 10^3$ lb./in^2 for steel

$= 2 \times 10^3$ lb./in^2 for timber

The possibility of using larger unit stresses and a larger modulus of elasticity in a bridge of this sort is worth considering. There is not the same danger in case of failure and the greatest stresses tend to become equal.

Substituting the above values we get :—

$q = 1{\cdot}67$
$d = 3{\cdot}9\ d^\circ$ ins with steel
$6{\cdot}5\ d^\circ$ ins with timber
$I = 5{\cdot}8\ I^\circ \times 10^4$ in^4 with steel
$= 8{\cdot}7\ I^\circ \times 10^5$ in^4 with timber
$M = 2{\cdot}7 \times 10^8$ in lbs, load uniform
$5{\cdot}4 \times 10^6$ in lbs, load concentrated

Suppose $l/c = 0{\cdot}1$.
Dead load thrust $= \frac{1}{2}$ raw $= \ldots\ldots$ 270,000 lbs.

Live load thrust $=$ rpa $\int$ h dz $=$ (see influence line)
$6 \times 1000 \times 150 \times 0{\cdot}075 =$ 67,000 lbs.

H= total thrust =337,000 lbs.

$I = a^2H/Ec^2 =$ (for timber) 5,400 in^4

M (uniform load) $= pa^2 \int m\,dz =$ (see influence line)
$1000 \times 150^2 \times 0{\cdot}009 =$ 202,000 ft. lbs.

M (single load)$= Wam = 3{,}000 \times 150 \times 0{\cdot}05 =$ 22,500 ft. lbs.

$d = 2\,f\,I/M = \ldots\ldots\ldots 8{\cdot}9$ ins.

Two beams 3′—9″ wide ×9″ deep give the necessary M of I.

The following table shows the effect of changing the depth :—

	d	l/c	I in^4	M. in-lbs.	
Steel	90″	0·93	39,500	$16{\cdot}7 \times 10^6$	2 light steel girders @ 75 lbs./ft. each
Timber	90″	0·72	312,000	$15{\cdot}9 \times 10^6$	2 timber girders, booms 50 in^2
Steel	36″	0·54	13, 00	$13{\cdot}6 \times 10^6$	2 plate girders @ 130 lbs. each
Timber	36″	0·33	82.400	$8{\cdot}9 \times 10^6$	2 timber girders, booms 80 in^2
Steel	12″	0·20	1,700	$5{\cdot}5 \times 10^6$	6 No. RSJs 12″×6″=260 lbs/ft.
Timber	12″	0·12	8,700	$3{\cdot}1 \times 10^6$	2 beams 2′—6″×12″=720 in^2
Steel	6″	0·09	230	$1{\cdot}9 \times 10^6$	6 No. 6″×5″=150 lbs.
Timber	6″	0·05	1,400	$0{\cdot}8 \times 10^6$	2 beams 3′—3″×6″=470 in^2

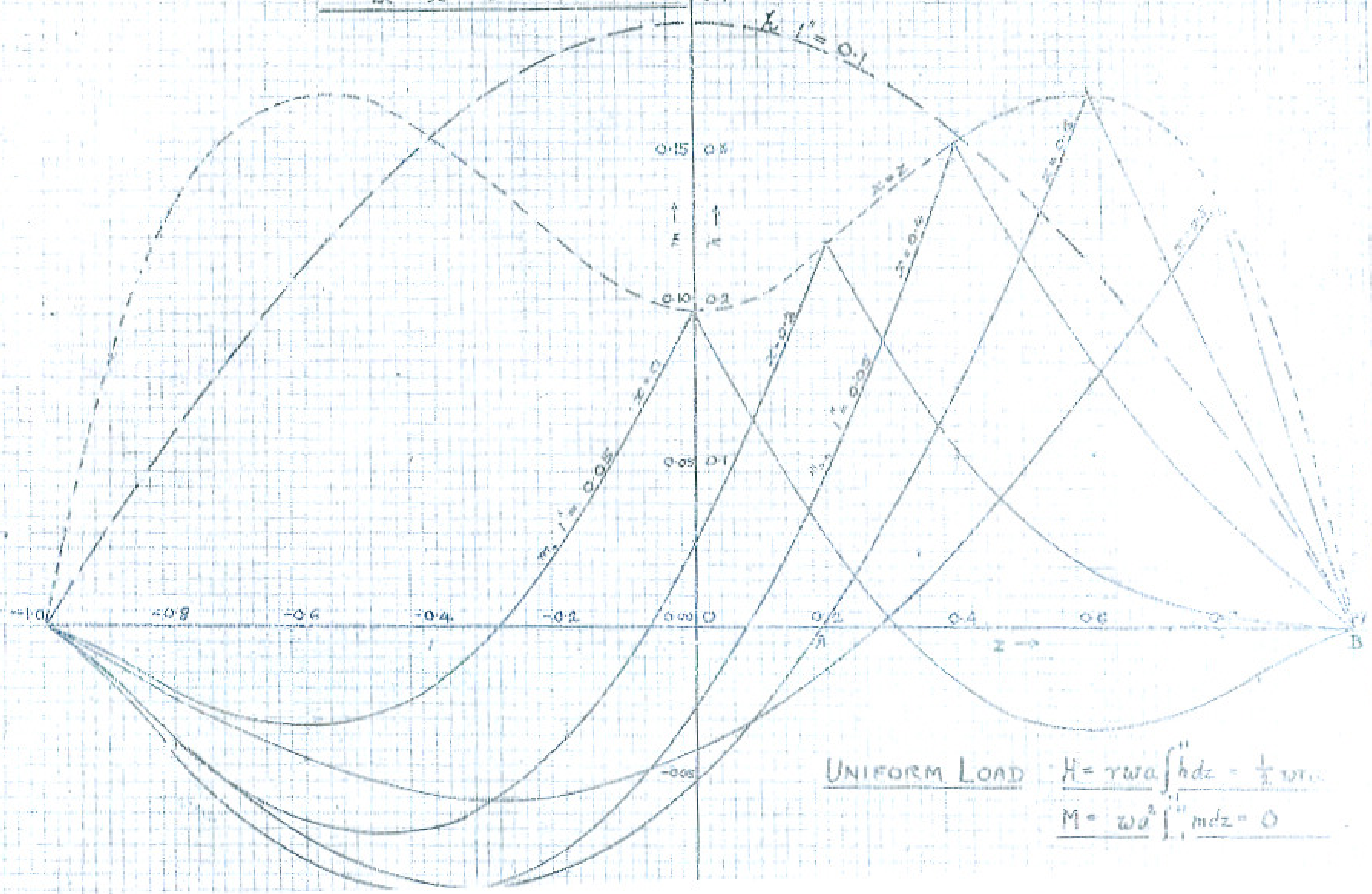
PLATE I

INFLUENCE LINES OF THRUST AND BENDING MOMENT $c = 0$

$H = Wrh \qquad h = \frac{5}{64}(5-z^2)(1-z^2)$

$M = Wam \qquad m = \frac{1}{2}(1-z)(1+x) - h(1-x^2) \quad x<z$

$= \frac{1}{2}(1+z)(1-x) - h(1-x^2) \quad x>z$



UNIFORM LOAD $\quad H = rwa\int_{-1}^{1} h\,dz = \frac{1}{2} wr\ldots$

$M = wa^2 \int_{-1}^{1} m\,dz = 0$

CORRECTION SLIP

TO

Mr. Lindley's Paper No. 97 on Canal Falls and their uses as Meters, read at the 1925 Annual Meeting.

Paper No. 97, page 89. *For* 3·90 *read* 3·09.

Plate II

Influence Lines of Bending Moment c = 6 etc.

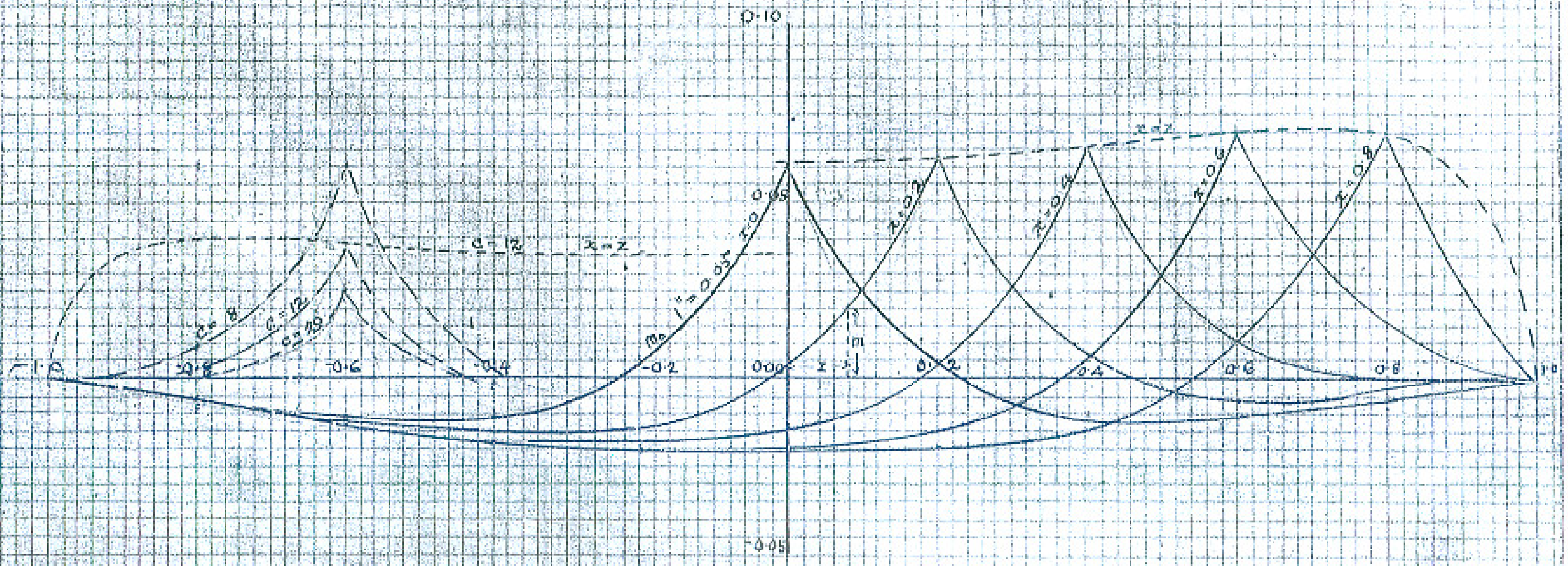
$x < z \qquad m = 0.000413\, e^{6x} \sinh 6(1-z) - 0.055h\left(1 - \frac{\cosh 6x}{201.7}\right)$

$x > z \qquad\;\; = 0.000413\, e^{-6x} \sinh 6(1+z) - 0.055h\left(1 - \frac{\cosh 6x}{201.7}\right)$

Approx. Formulae:–

$m = \frac{1}{2c}\left[e^{-c(z-x)} - \frac{4h}{c}\right] \qquad x < z$

$\;\;\; = \frac{1}{2c}\left[e^{-c(x-z)} - \frac{4h}{c}\right] \qquad x > z$



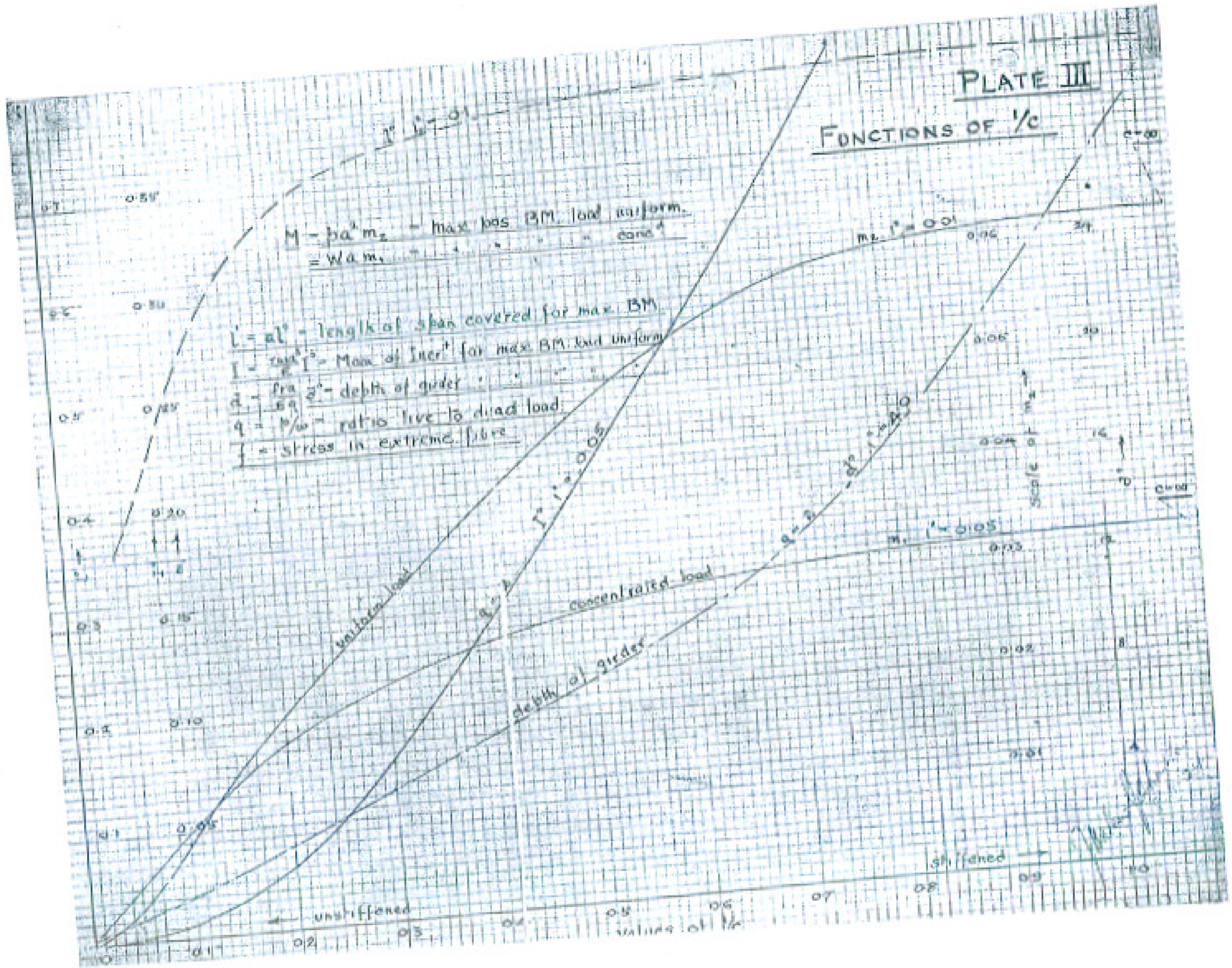


PLATE III
FUNCTIONS OF 1/c
uniform load
concentrated load
depth of girder
unstiffened
stiffened
Values of 1/c

DISCUSSION.

The Author introducing his paper said he had attempted to make possible the design of a suspension bridge without stiffening girders and yet possessing a certain degree of rigidity; to fill, in other words, the gap between the stiffened and the unstiffened bridges. He felt an apology was due from him for presenting the Congress with a mathematical paper but he could hardly give the paper without the mathematics and as the subject as far as he was aware was new there were very few examples of this type of bridge which could be used as illustrations. His first introduction to the subject was due to Mr. Astbury, Chief Engineer, P. W. D., who designed a suspension bridge with stiffened floor. Mr. Lyster afterwards designed a similar bridge. He regretted the latter was not present to give his views on the subject. On the destruction of the first Pandoh bridge the author was asked to design the new one. From the experience gained in the first bridge he decided to use two stiffening beams 15 inches by 10 inches in place of the original girders. At that time he did not know how to calculate the Bending Moments and the beams were made very similar to the bottom booms in the first bridge. When these booms had been put in place the bridge had been tested and seems to have all the stiffness necessary. If the calculations given in the paper were now applied to these beams it would be found that they were on the weak side. The loading assumed was a dense crowd of people covering a portion only of the span as shown in Plate II. The chance of such a load at present was very unlikely and the beams were doing their duty quite satisfactorily.

But the main object of the paper was if possible to do away with the absurd practice of using large stiffening girders in bridges on hill roads in India. These girders were very expensive and they were very seldom made strong enough; the booms were of short timbers and the joints had an efficiency of under 5 per cent. The first load on the bridge destroyed these joints and the result was an unsightly structure. Two examples of the kind could be seen at Mandi and Oot. The first Pandoh bridge 230 feet span had booms made up of three lines of 12 feet railway sleepers and in all 7,000 bolts 1 inch by 15 inches were required to develop a tension equivalent to two sleepers. If the money spent on these bolts had been spent instead on *surkhi*, the bridge would still be standing.

The author thanked the S cretary for the care which had been taken in printing the paper. But there were a few corrections due to mistak s in the original manuscript, which he drew attention to.

Mr. A. R. Astbury said that he wished to congratulate Mr. Johnston or writing a paper for the Congress on such an intricate subject. The roblem was exceptionally difficult and involved a lot of mathematical rk.

The President said he was afraid it was too much to expect ordinary ividuals to cross swords with the author but perhaps some of the athematicians might deal with some of the points raised by corresondence later.