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ENERGY THEORY OF TURBULENT FLOW OF LIQUIDS.

By T. Blench, B.Sc.(Hons.), A.M.I.C.E., I.S.E.

1. The object.

The object of this paper is to find, by using Hamilton's Principle, the Energy Relations behind authoritative flow formulae and thereby establish a new flow formula fo "rough" conditions of rigid channels and pipes. Incidentally a further meaning is given to certain quantities whose exact nature is not known.

2. Introduction.

To prevent the analysis seeming indirect and involved it is advisable to show first how by using clues from its development and results existing information can be corrected and corelated.

(*i*) *Old Flow Formulae.* The Chezy Equation $V=C\sqrt{RS}$ has always been, and still is the basis of flow formulae investigation. Before our present knowledge of boundary layers there seems to have been no appreciation of the apparent difference between smooth boundary and rough boundary turbulence, except by physicists, and the result was "universal" formulae of the Kutter type for C, which are distinguished by dimensional heterogeneity that renders them of little use as dynamical pointers. Bazin's formula is an improvement in that it can be rendered dimensionally homogeneous by giving his γ the dimensions of $L^{\frac{1}{2}}$; but it is not suggestive. Mannings' formula, however, is most suggestive and its simplicity suggests that it is based on one class of data.

It is :—

$$V=\text{const.}\ R^{\frac{2}{3}}\ S^{\frac{1}{2}}\ldots\ldots\ldots\ldots(1)$$

where R is hydraulic mean depth and S is surface slope for a channel and non-dimensional pressure gradient for a pipe.

Smooth boundary turbulence gave the formula

$$V=\text{const.}\ R^{\frac{5}{7}}\ S^{\frac{4}{7}}\ldots\ldots\ldots\ldots(2)$$

to Woltmann (1804) and Flamant (1892) (Bib. 1, page 41) and was merely dimensionally corrected by Blasius (1911) to

$$V = \text{absolute const. } (N)^{\frac{1}{8}} \sqrt{gRS} \ldots\ldots(3)$$

where N is Reynold's Number.

(*ii*) *The Influence of Reynolds* was to bring the physicist's investigations on to dimensional lines; but it did not affect engineers' formulae noticeably.

(*iii*) *More modern investigation* has been on velocity distribution associated with the names of Prandtl, Nikuradse, von Karman, Stanton and others. It has produced nothing final in the way of flow formulae, but has given a wealth of valuable information and ideas. The more important, for this paper,, are :—

(*a*) *Relative Roughness.* Writing conventionally,

$$V = \sqrt{8gRS/\lambda} \ldots\ldots\ldots\ldots(4)$$

Blasius (1911) (Bib. 1. page 44) pointed out that λ, when the formula applied to rough flow, must depend, not merely on the boundary protuberance size, l, but on l/R, the " relative roughness".

(*b*) *Boundary Layer.* Instead of there being slip at the boundary there is a layer of small thickness δ, in which viscosity has a preponderating effect. The order of the relative boundary layer thickness δ/R is that of $\sqrt{1/N}$, and in this layer the boundary velocity drops rapidly to zero.

(*c*) *Rough and Smooth Boundaries.* These terms are now understood to be relative to the flow. When δ is sufficiently great to mask the boundary protuberances of height l the boundary is smooth and the flow formula contains viscosity, but not l. When δ is, or would be, much less than l, then the protuberances break the layer, and the boundary is rough. The flow formula then contains l but not ν. This is the commonest case in engineering. Between the two stages is a transition not susceptible to exact analysis, this stage being the one which confuses purely empirical work.

(*d*) *Mixing Velocity, V_*.* This (Bib. 2) appears to be due to Prandtl recently, although the ultimate idea must go back to at least the Kinetic Theory of Gases. He imagines the frictional resistance of the fluid to arise from transference of momentum between layers by the passage of particles. The velocity of transference is of the order of

$V_* = \sqrt{\frac{\tau}{\rho}}$, where τ is the boundary shear stress. He calls V_* the " shear stress velocity"; but this article will use the more elegant term" Mixing Velocity" as it is not required in any other connection.

(*iv*) *Prandtl's Flow Equations.* The account of these is in Bib. 2, and they are associated with several men's work. They are

$$V=\sqrt{8gRS}\ (2{\cdot}0\ \log_{10}\ 2R/l+1{\cdot}74)\ldots\ldots..(5)$$

for rough flow, and (4) for smooth flow with

$$\frac{1}{\sqrt{\lambda}}=2\ \log_{10}\ N\sqrt{\lambda}-0{\cdot}8\ldots\ldots\ldots(6)$$

These equations are said (Bib. 2) to fit Nikuradse's experiments. (5) is tested in Bib. 2, Fig. 9 against Nikuradse's data for artificially sand-coated circular pipes of 2·5, 5 and 10 cm. diameter with $2R/l$ varying from 15 to 252—the whole range being up to 507. (6) is stated to be fitted imperfectly.

These formulae are based on the sound, but not very explicit idea of mixing velocity, and on certain plausible assumptions, and devised to fit the fact that a logarithmic curve fits both the boundary layer (whose exact shape is doubtful), and the inner fluid velocity distributions with considerable accuracy. The more obvious defects of the underlying theory are that it gives the velocity distribution curve a cusp on the pipe centre line, and the derived constants have to be altered slightly to fit facts.

(5), by virtue of its derivation, is of dynamically sound form a good fit to data and it *contains relative roughness explicitly.*

(*v*) *Manning's Formula Dimensionally.* (5) suggests dimensional analysis of (1), resulting in :—

$$V=\text{absolute const.}\ (R/l)^{\frac{1}{6}}\ \sqrt{gRS}\ldots\ldots\ldots\ldots(1')$$

which shows that Manning's formula is for rough flow, and contains all the physical implications of Prandtl's.

Fig. 2 compares $(R/l)^{\frac{1}{6}}$ with $1/\sqrt{\lambda}$ of (5) over the range of Bib. 2. Fig. (9) and over the whole range of Nikuradse's experiments and it will be seen that there is practically no difference in the formulae as a fit to data.

(*vi*) *The Author's Formula* for rough flow is

$$V=\text{const.}\ \ R^{\frac{3}{4}}\ S^{\frac{1}{2}}\ldots\ldots\ldots\ldots(7)$$

$$\text{or}\quad V=\text{absolute const.}\ (R/l)^{\frac{1}{4}}\ \sqrt{gRS}\ldots\ldots\ldots(7')$$

Fig. 1 shows the most suitable data from Barnes' "Hydraulic Flow Reviewed" testing the formula, and Fig. 1 (*a*) shows the range of R and S covered, illustrating incidentally, the practical difficulty of obtaining a wide variation of R without having to change the roughness.

Manning's or Prandtl's formula would also fit the data excellently. Fig. 3 shows (7′) compared with (5) in the manner of Fig. 2. The formula stands the test.

(*vii*) *Author's Formula adapted for Smooth Flow.* Assuming (7′) correct it is reasonable to expect that, for smooth flow, R/l would be replaced by R/δ, i.e., by $\sqrt{N}$, making the smooth flow formula

$$V = \text{absolute constant. } N^{\frac{1}{8}} \sqrt{gRS} \ldots\ldots\ldots (8)$$

But this is precisely Blasius' Formula (3) for smooth pipes, so the formula (7) receives strong confirmation, while Manning's formula appears as a very close approximation to truth ; and the flow formulae for smooth and rough boundaries, apparently so different, are both derived from the same fundamental law.

(*viii*). *Incoherent Boundaries. The Lacey Theory.* Channels such as canals in alluvial plains, and rivers in spate with the bed material active form a special class. They are characterized by the fact that they form their own section, shape and slope in their own incoherent transported material. A rigid channel can only adjust its R; a rigid pipe its S; but a Regime Channel adjusts its R, S and P/R, P being wetted perimeter. These terms are all uniquely determinable as functions of the discharge Q and the Lacey Silt Factor f. The whole theory is detailed in the Institution of Civil Engineers' Papers 4736 and 4893.

(*a*). As might be expected the Lacey Formulae contain a standard type flow equation :—

$$V = \text{const. } R^{\frac{3}{4}} \; S^{\frac{1}{2}} \ldots\ldots\ldots\ldots (9)$$

$$= \text{abs. const.} \left(\frac{R}{\frac{f_1 V_L^2}{g^2 c^2}} \right) \sqrt{gRS} \ldots\ldots\ldots\ldots (9')$$

in exact correspondence with (7) and (7′) but the "equivalent protuberance" contains viscosity in the term

$$V_L = (\nu g)^{\frac{1}{3}} \ldots\ldots\ldots\ldots (10.)$$

(9) corresponds to the adjustable R.

Corresponding to the adjustability of S, another equation,

$$V = 16 \sqrt[3]{R^2 S} \ldots\ldots\ldots\ldots (11)$$

$$\text{or } V^3 = (Cg^2/V_L)\, R^2 S \ldots\ldots\ldots\ldots (11')$$

holds.

Corresponding to the adjustability of P/R the equation

$$P/R = 7{\cdot}12V \ldots\ldots\ldots\ldots (12)$$
$$= C'V/V_L \ldots\ldots\ldots\ldots (12')$$

holds. C and C' are both non-dimensional, but may contain silt relative density.

Combination of (9) and (11) gives

$$f_1 = V^2/R \ldots\ldots\ldots\ldots (13)$$

f_1 being a proportional to the Lacey Silt Factor f, and $\propto m\sqrt{gd}.g/(V_L \rho')$, d being mean silt particle diameter and m the mass of silt per second per cusec, or a function of it.

(*b*) $V^2/R = f_1$ is referred to by Lacey as a "turbulence criterion". It will be found that it comes into all energy expressions in the author's theory. It is the Lacey Theory which, by finding that f_1 defines "silt load," has brought it into prominence as a new concept in general turbulence theory.

(*c*) $V_L = (\nu g)^{\frac{1}{3}}$ is a further new concept arising from the Lacey Theory. It has a definite correspondence with V_*, and appears to be a mixing velocity related to the sand-water layer.

(*d*) The existence of a second flow formula (11) obviously indicates the dual nature of the boundary layer (the same idea occurs in Bib. 1, page 78 for clean water). The ordinary boundary layer is modified by the presence of the active incoherent bed material.

Prandtl has done for pipes what Lacey has done for Regime Channels. Both have approached their subject with a dynamical outlook. Prandtl has, however, worked from velocity distribution, and owes much to others. Lacey has worked from flow data, and his data is colossal in range and quantity, so his results have an authority denied to empirical rigid channel formulae, whether that empiricism be of directly determining flow formulae, or of fitting curves to velocity distribution data, and deriving the flow formulae.

The Lacey formulae have been accepted by the Central Board of Irrigation of the Government of India for design.

(*ix*). *Correlation of information.* The practical use for several years, and theoretical consideration of the Lacey Theory, particularly the implication of Lacey's dimensional ideas and his V^2/R, suggested the energy line of investigation to the writer. This produced his formula (7) which, along with the idea of relative roughness and Prandtl's formula suggested the mean of writing the flow formulae for all cases, i.e., (1), (7) and (9) in their dimensionally correct form, and correcting (1) to (7). This, and analysis, suggested that the smooth boundary formula (2) must be of the same form, viz., (3) where $\sqrt{N}$ is given its proper meaning.

Prandtl's mixing velocity idea suggested linking V_L with it. The complete linkage of results previously considered unconnected should assist the study of the mechanism of turbulence.

(*x*). *The Mechanism of Turbulence* is illustrated in the dirty water of irrigation canals. The surface is observed to be continually broken by *rosettes* of fine silt at short space intervals. This indicates that momentum transference is effected by vortex rings formed at the bed, and rising to the surface. This phenomenon adds definiteness to Prandtl's idea of mixing velocity, referring it not to particles (which would appear legitimate only in gases) but to eddies.

3. Hamilton's Principle.

Consider the system of particles which constitute the discharge Q of a pipe or channel. Were there no internal friction the particles would adjust themselves to such a configuration that $\delta\,(\overline{L}-\overline{V})=0$, where $\overline{L}$ is the K. E. and $\overline{V}$ the Potential Energy. The effect of internal friction is exactly to balance $\overline{V}$, and the condition becomes $\delta\,\overline{L}=0$, i.e., the K. E. of flow must be a minimum.

Now the velocity at any point in the fluid is the mean velocity plus a small deviation which averages zero; so the K. E. per lb. at the point is the K. E. of the mean motion, plus the K. E. of the relative motion, and if the deviation of velocity is considered as random the principle of least squares shows that the K. E. of the relative motion must be a minimum.

Summing throughout the system comprising Q, and knowing that the K. E. is a minimum, and so is the K. E. of the relative motion, it follows that the K. E. of the mean motion must be a minimum, *i.e*, QV must be a minimum. But as we consider fixed discharge Q it follows that,

$$V \text{ must be a minimum} \ldots\ldots\ldots\ldots (14)$$

Strictly V is root mean square velocity which, in practical cases does not differ appreciably from the mean velocity V.

The condition (14) means, for a rigid channel (which can only adjust R) that $dV/dR=0$; for a rigid pipe $dV/dS=0$; and for a Lacey Regime Channel $dV/dR=0=dV/dS$.

4. The Energy Dissipation Equation.

Using the old idea of boundary slip the equation for the fluid contained between two cross-sections unit distance apart will be "Rate at which Gravity works equals Rate of working of Boundary Resistance plus Internal Rate of Dissipation of Energy".

The modification to fit modern ideas is verbal, *viz.* "Rate at which gravity works equals Rate of Transformation of Energy at the Boundary Layer, plus Rate of Dissipation within the fluid." And for "Boundary Resistance is quadratic in V_b" we say "Rate of energy transformation is cubic in V_b".

Our problem is to give symbolic expression to the energy equation so that, on treatment by the Minimum Velocity Condition, known, or probable, flow equations result.

(*i*) *Rigid Rough Pipe.* Consider a pipe of smooth cross-sectional shape, so that the perimetral velocity V_b shall be constant. (This excludes rectangular pipes, which are unsuitable for experimental determination of the laws of flow). V_b is the "slip velocity" of older theory or to fit boundary layer theory, it is very nearly the velocity at the boundary found by extrapolating from the velocity distribution curve of the inner fluid before it comes under the effect of the boundary layer.

All experiments suggest that the resistance is quadratic in V_b. The idea of relative roughness suggests that it must vary inversely as some power of R. Now consider two pipes of different R, but the same absolute roughness. The larger will have a relatively less protuberance height, but relatively more protuberances per unit area of boundary. Therefore the power of R is unlikely to be the first, and is very probably $\frac{1}{2}$. This brings the boundary resistance per unit area to the form $\Phi V_b^2/R^{\frac{1}{2}}$. As it must depend on the fluid density ρ and protuberance height l, dimensional considerations show that it must further reduce to absolute constant $\rho\,(l/R)^{\frac{1}{2}} V_b^2$. Viscosity cannot enter.

The boundary resistance equation for rough flow is therefore

$$\rho g P R S = \Phi P V_b^2/R^{\frac{1}{2}} \qquad (15)$$

$$= \text{abs. const. } P\rho(l/R)^{\frac{1}{2}} V_b^2 \qquad (15')$$

giving

$$V_b^2 = (\rho g/\Phi) R^{\frac{3}{2}} S \qquad (16)$$

$$= \text{abs. const. } (R/l)^{\frac{1}{2}} gRS \qquad (16')$$

Considering unit length the Rate of Boundary Work is $\Phi P V^3_b / R^{\frac{1}{2}}$, which, writing Q/VR for P, and simplifying, reduces to

$$\text{Rate of Boundary Work} = Q\Phi(\rho g/\Phi)^{\frac{3}{2}} R^{\frac{3}{4}} S^{\frac{3}{2}}/V. \qquad (17)$$

The rate of internal working must be, at present, conjectural. It must vary as Q and as some power of V. It must further depend in some way on how velocity or turbulence is distributed, i. e., on some average velocity gradient such as V/R or on the Lacey turbulence criterion V^2/R. Finally it will depend on some or all the properties of the fluid and boundary, denoted by k. (We need not anticipate that k will not contain viscosity). In fact, whatever view we take, it should be possible to write,

$$\text{Rate of Internal Work} = kQV^m/R^n \qquad (18)$$

The rate of Gravity Work is $\rho g Q S$ (19)

Equate (19) to (17) plus (18) and divide by $\rho g Q$, getting

$$S = (\rho g/\Phi)^{\frac{1}{2}} R^{\frac{3}{4}} S^{\frac{3}{2}}/V + (k/\rho g) V^m / R^n \qquad (20)$$

which represents the Energy Dissipation Equation.

Multiplying both sides by gV will give rate of working per lb. of fluid.

Applying the condition $dV/dS=0$(rigid pipe) to (20) we find that the doubtful second term on the right does not affect the answer which is

$$V = 3/2(\rho g/\Phi)^{\frac{1}{2}} R^{\frac{3}{4}} S^{\frac{1}{2}} \qquad (21)$$

$$= 3/2(R/cl)^{\frac{1}{4}} \sqrt{gRS} \qquad (21')$$

using the value of Φ found dimensionally for (15′). The term c is introduced so that the constant 3/2 can be retained whether we take protuberance height relative to R, or r, or any other linear dimension of the pipe.

(21′) with (15′) gives

$$V_b = 2/3V \qquad (22)$$

(*a*) This *Boundary Velocity Ratio* is confirmed from Figs. 25, 28 and 31(II) of Bib. 1 (Prandtl) by drawing tangents from $V_b/V = \frac{2}{3}$ to the velocity distribution curves, remembering that although the curves bend rapidly near the boundary the boundary layer is confined, in its full effect, to about 1/100th of the radius.

(*b*) The *Flow Equation* were it not for further information, might be considered as derived from a $L/R^{1/r}$ resistance law. Repetition of the preceding analysis would then show that the Boundary Velocity Ratio is still $\frac{2}{3}$; but the index $\frac{1}{4}$ in the Flow Equation would be replaced by $\frac{1}{2r}$. The only likely value of r besides the 2 chosen by the writer is 3, and this would give Manning's Equation.

(*ii*) *Rigid Rough Channels.* Channels must obey the Pipe Law. For, if they did not, imagine a circular conduit running exactly half full. By symmetry we should expect it to obey the same law when just exactly full. This latter condition, however, is the limiting case of pipe flow, so we should expect the Pipe and Channel Laws to give the same result for this condition. This coincidence must apply no matter what the R and S, so can only occur if the laws are identical.

We may, therefore, find m, n and k of (20) by applying the condition $dV/dR=0$, and arranging m, n and k to give (21).

A simpler method is to insert the value of S from (21) in the first member of the right hand side of (20) which will reduce it to $(\frac{2}{3})S$ making the second member $(\frac{1}{3})S$. Use the value of $1/S$ from (21) getting the identity,

$$(k/\rho g)V^{m}/R^{n}=\tfrac{1}{3}V^{2}\,\tfrac{4}{9}(\Phi/\rho g)\,(1/R)^{\frac{3}{2}}$$

which can only be satisfied if $m=2$, $n=3/2$, $k=4\Phi/27$.

(*iii*) *Rough Channels and Pipes* then, have the same flow equation, and the same Boundary Velocity Ratio 2 : 3, divide their Boundary and Inner Rates of Working in the ratio 2 : 1 and have the same energy dissipation equation

$$S=(\rho g/\Phi)^{\frac{1}{2}}R^{\frac{3}{4}}S^{\frac{3}{2}}/V+4/27(\Phi/\rho g)V^{3}/R^{\frac{3}{2}} \tag{23}$$

$$=(R/cl)^{\frac{1}{4}}g^{\frac{1}{2}}R^{\frac{1}{2}}S^{\frac{3}{2}}/V+4/27(cl/R)^{\frac{1}{2}}(V^{2}/gR) \tag{23'}$$

The rate of internal working per pound is

$$4/27\ (cl/R)^{\frac{1}{2}}\ (V^{2}/R)V \tag{24}$$

This, combined with Sec. 2 (x) suggests that Lacey's turbulence criterion V^2/R measures the mean internal resistance per lb. of fluid to the upward moving eddies, that V measures their mean speed, and that cl/R is the reducing factor that brings the measure V to its final value.

(*iv*) *Lacey's Turbulence Criterion* is thus a vital link between incoherent and rigid boundary theories.

(*v*) *Prandtl's* V_*

$$V_* = \sqrt{\frac{\tau}{\rho}} = \sqrt{(4V^2/9)\ (\Phi/\rho R^{\frac{1}{2}})}$$

$$= 2/3\ \ V\,(cl/R)^{\frac{1}{4}} \qquad (25)$$

For a 24" pipe of concrete with projections of say 0·04" $V=3$ feet / sec and $c=\frac{1}{2}$ (i. e. relating protuberance to radius)

$$V_* = 0{\cdot}5 \text{ ft./sec.}$$

This supports the view that V_* measures *eddy* speed. Using (25) in (24) we find,

$$\text{Rate of internal work per lb.} = \tfrac{1}{3}(V_*^2/R)\ V. \qquad (24')$$

$$= \tfrac{1}{2}(V_*^2/R)\ \ V_*\ \ (R/cl)^{\frac{1}{4}}. \qquad (24'')$$

As we know that V_* is at right angles to the flow this confirms the view on the meaning of V^2/R, and suggests how the idea of V_* might be improved to explain the mechanism of turbulence.

(*vi*) *Smooth Pipes and Channels.* As explained in 2 (*b*) Boundary Layer Theory immediately suggests that the smooth pipe analysis differs from the rough merely in that $R/\delta \propto \sqrt{N}$ replaces R/l throughout.

The resistance law will be

$$\text{Resistance} = \text{abs. const.}\ \ P\rho V_b^2/(VR/\nu)^{\frac{1}{4}} \qquad (15'')$$

and, repeating the energy analysis, it will be found that

$$V_b = 7/11\,V \qquad (22'')$$

$$V = \text{abs. const.}\ \ N^{\frac{1}{8}}\sqrt{gRS} \qquad (21'')$$

Rate of working per lb. internally

$$= \text{abs. const.}\ (1/N)^{\frac{1}{4}}\ (V^2/R)\ V \qquad (24'')$$

$$V_* \propto 2/3.\ V\ (1/N)^{\frac{1}{8}} \qquad (25'')$$

Smooth Pipe and Channel Theory is, thus, derivable from the same energy law as for rough conditions and the internal turbulence mechanism is the same.

(*viii*) *Lacey Channels.*

(*a*) Write Laceys' flow formula (9') as

$$V = 1{\cdot}5\left(\frac{R}{\dfrac{3^4 f_1\ V_L^2}{2^4 g^2 c^2}}\right)^{\frac{1}{4}}\sqrt{gRS} \qquad (26)$$

$$= 1{\cdot}5\ \ (R/c't)^{\frac{1}{4}}\sqrt{gRS} \qquad (26')$$

This relates it exactly to the rough flow rigid channel formula, and t is now "equivalent protuberance."

The value of f_1 given in 2.(*viii*), *viz.*, $f_1 \propto m\sqrt{gd}.\ g/(V_L\rho')$ is obtained by dimensional analysis from Lacey's result that $f_1 \propto d^{\frac{1}{2}}$. It *may*, therefore, be subject to slight improvement. This does not affect his flow equations in which f_1 is V^2/R. In fact the $f_1 \propto d^{\frac{1}{2}}$ relation is an attempt to link the Lacey Theory with the physical properties of the boundary layer and fluid, and is extraneous to the self-contained theory linking V with R and S.

To proceed, it is obvious that (26) must be derivable from exactly the same form of energy equation as for rigid channels, and the same form of resistance law, viz., an impact one. It must, therefore, be derived by taking as boundary the bottom of the active incoherent layer where the sand will be stationary. The impact will, however, be, not from clean water, but from a sand-water mixture. The nature of the impact will depend on the proportion of sand to water, and how that sand is distributed in the sand-water layer, i.e., it will almost certainly depend on viscosity. Further, the fact that the sand and water are of different densities may be expected to bring g into the formula.

Writing $t \propto (f_1 V_L{}^2 g^2 C^2) \propto (m\sqrt{gd}\ V_L/\rho' g C^2)$ (27)

it is seen that the factors to be reasonably expected do occur. ρ' must be related to both water and sand density; but in a practical formula ρ will suffice.

(*b*). The Regime Test Formula (11′) rewritten as equation (28)

$$V^3 = (Cg^2/V_L)R^2S \qquad (28)$$

is also a flow formula.

It is easy to work inversely to find from what energy equation, and resistance law this can be deduced. The result is,

$$\text{Resistance} = \Phi' P(V_b{}')^3/R \qquad (29)$$

$$V = \tfrac{4}{3}.\ (\rho g/\Phi^1)^{\frac{1}{3}} R^{\frac{2}{3}} S^{\frac{1}{3}} \qquad (30)$$

$$V'_b = \tfrac{3}{4}V \qquad (31)$$

and the complete energy equation is

$$S = \tfrac{3}{4}(Cg^2V_L)^{\frac{1}{3}} R^{\frac{2}{3}} S^{\frac{4}{3}}/V + (V_L/4Cg^2)\ (V^3/R^2) \qquad (32)$$

Comparison of (30) with (28) gives

$$\Phi' = \rho V_l/Cg \qquad (33)$$

which throws the curious resistance law back into a standard form.

$$\text{Resistance} = \text{abs. const.} \quad P\rho\,(V_b'^2)\,(c't/R)^{\frac{1}{2}} \qquad (33a)$$

by using $f_1 = V^2/R$, and $V'_b = \frac{3}{4}V$.

The boundary, and internal rates of working are now $\frac{3}{4}$ and $\frac{1}{4}$ of the whole, against $\frac{2}{3}$ and $\frac{1}{3}$ of the rigid boundary case, or the case represented by (26).

(c). *Implication of the Two Flow Formulae.* Fig. 4 reconciles the two results. Above and through the Prandtl layer, shown to a very exaggerated scale, must be another layer of sand-water mixture in which the sand concentration is so great as to make the behaviour much different from that of the inner fluid. If the silt load is all of one grade it all occupies the dual layer, whose upper portion we shall, for convenience, call the Lacey Layer. If the silt is, as usually in practice, of mixed grade the heavy grade occupies the dual layer, and the fine material is in suspension in the body of the fluid but not, in the problems we consider, in sufficient quantity to make the behaviour appreciably different from that of pure water. The bottom of the Prandtl Layer is stationary.

Exactly as in the rigid channel case the Prandtl layer is bounded by the motionless bed, and the stratum of mean velocity $\frac{2}{3}V$, and accounts for $\frac{2}{3}$ of the rate of working by gravity. In the inner fluid the remaining one third is accounted for; but between the Prandtl Layer and the relatively pure inner fluid is now the Lacey Layer, bounded by the mean speeds $\frac{2}{3}V$ and $\frac{3}{4}V$ which abstracts 1/12 of the total energy leaving only $\frac{1}{4}$ for the real inner fluid.

One of the two flow equations arises according as we start from the "Resistance Law" or, more correctly, as has been pointed out already, the "Energy Dissipation Law" of the Prandtl layer or the Prandtl-Lacey Layer respectively.

(*d*) *Rate of Working.* Using (24), with $c't$ replacing cl, and multiplying by 3 we get :

$$\text{Rate of working by Gravity per lb.} = V_L f_1^2/gc. \qquad (34)$$

This shows that, for given physical conditions, a Regime Channel adjusts itself so that the rate of working by gravity per lb. is the same, *whatever the discharge.* This law helps to explain the tortuosity of rivers in low stage in alluvial plains.

(34) Can be thrown into the standard form of "(turbulence criterion)×(a velocity)×(a fac or)".

Equation (25) adapted, gives

$$V_*^2 = VV_L\, f_1/(Cg) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (35)$$

Whence (27) gives

$$V_* = V(m/\beta')\sqrt{gd/C} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (36)$$

and (24') leads to a rate of working by gravity of

$$(V^2/R)\sqrt{gd}.\ m/\rho'\sqrt{C}\ldots\ldots\ldots\ldots\ldots(37)$$

It is not possible, without using the shape formula (12), to obtain a similar form to (24'), in terms of V_L and direct physical constants (unmixed with f_1). The result from (34) is then

$$(V_L^2/R)V\,(P/R)^2(1/gCC'^2)\qquad\ldots\ldots(34'').$$

(*d*) Prandtls' V_* appertains to the boundary, and the function (cl/R) or $(c't/R)$ looks after the distribution of mixing velocity throughout the fluid.

(*e*) V_L remains the most difficult concept to elucidate.

The definition of "equivalent protuberance", can be written

$$(V^2/R)\ (V_L^2/t) \propto g^2\ C^2\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

and seems to make V_L a mixing velocity associated with the Lacey Layer in the same way that V_* is associated with the Prandtl Layer. But whereas (35) shows that V_* increases with V and f_1, V_L is $(\nu g)^{\frac{1}{3}}$ and fixed. Yet (34'') adds confirmation to the mixing velocity idea.

The definition of V_L suggests that it is a measure of the slip velocity required to keep a particle of given size just in the state of incipient suspension requisite for its transport. ν measures the lifting force, and g the resisting force. It follows that V_L is always associated with a constant C, or C' [equations (11) and (12)] which, although non-dimensional, is a function of the difference in relative densities of silt and water, so that when there is no silt (i. e. water has to lift water) the effect of V_L ceases.

(34'') suggests, then, that the silt layer adjusts itself so as to let a mixing velocity V_L be impressed on the boundary of the inner fluid. The boundary has then to adjust itself, according to the Shape Formula (12') so that the average mixing velocity throughout the fluid may be adequate to dissipate energy at $\frac{1}{4}$ the rate at which gravity works. In other words, instead of the mixing velocity varying with V as in a rigid channel, so that the energy can be dissipated according to the energy equation, the mixing velocity is limited by the Lacey Layer and the shape has to change, the ratio P/R increasing with V.

(*f*) The suggestions of this paragraph (*e*) are admittedly conjectural; but are given as the subject is very important for the next stage of the Lacey Theory, viz., the study of the mechanism of silt transport.

5. Conclusions.

(*a*) By means of Hamilton's Principle, which leads to an Energy Dissipation Equation of Turbulent Flow, the older empirical formulae of Rough Flow and Prandtl's equations of Rough and Smooth Flow can be corrected and linked with the correct formula of Blasius for Smooth Flow, and Lacey's correct formulae for Incoherent Boundaries.

(*b*) The general equation of flow is

$$V = \text{abs. const.} \left(\frac{R}{t_r}\right)^{\frac{1}{4}} \sqrt{gRS}.$$

where $\frac{t_r}{R}$ is relative thickness of the "brake" at the boundary, i. e.,

(*i*) It is l/R where l is protuberance height when conditions are "rough rigid".

(*ii*) It is δ/R where δ is boundary layer thickness when δ is $> l$ i. e. when conditions are "smooth rigid"; and then

$$\delta/R \text{ is } \propto (\nu/VR)^{\frac{1}{2}} \propto 1/N^{\frac{1}{2}}$$

(*iii*) It is t/R where t is "equivalent protuberance" when channel is formed in its own incoherent transported material.

(*c*) All formulae for turbulent flow are derivable from the one form of Energy Equation.

(*d*) Prandtl's Mixing Velocity is a measure of boundary eddy velocity.

(*e*) Lacey's Turbulence Criterion is common to all turbulent flows, and is a measure of resistance to eddy motion per lb. of fluid.

(*f*) Lacey channels are distinguished by a special sand-water layer above and through the Prandtl Layer.

(*g*) The perimetral velocity bears a fixed ratio to the mean for each sub-type of turbulent flow.

(*h*) The meaning of $V_L = (\nu g)^{\frac{1}{3}}$ is not clear; but it appears to be a mixing velocity imposed by the Lacey Boundary Layer, and explicable in terms of the mechanism of silt transport.

(*i*) Internal and Boundary Rates of working are adjusted in a fixed ratio for each sub-type of turbulent flow.

APPENDIX.

Values of C in $V = CR^{\frac{3}{4}} S^{\frac{1}{2}}$

(Using data of Barnes' Hydraulic Flow Reviewed)

Material	C	Barnes Table
Rock Faced Masonry in cement ..	89	XIV
Clean Hard-brick well pointed conduits	115	XI
Dressed Masonry cement	123	XIII
Clean Neat Cement Pipes	158	X

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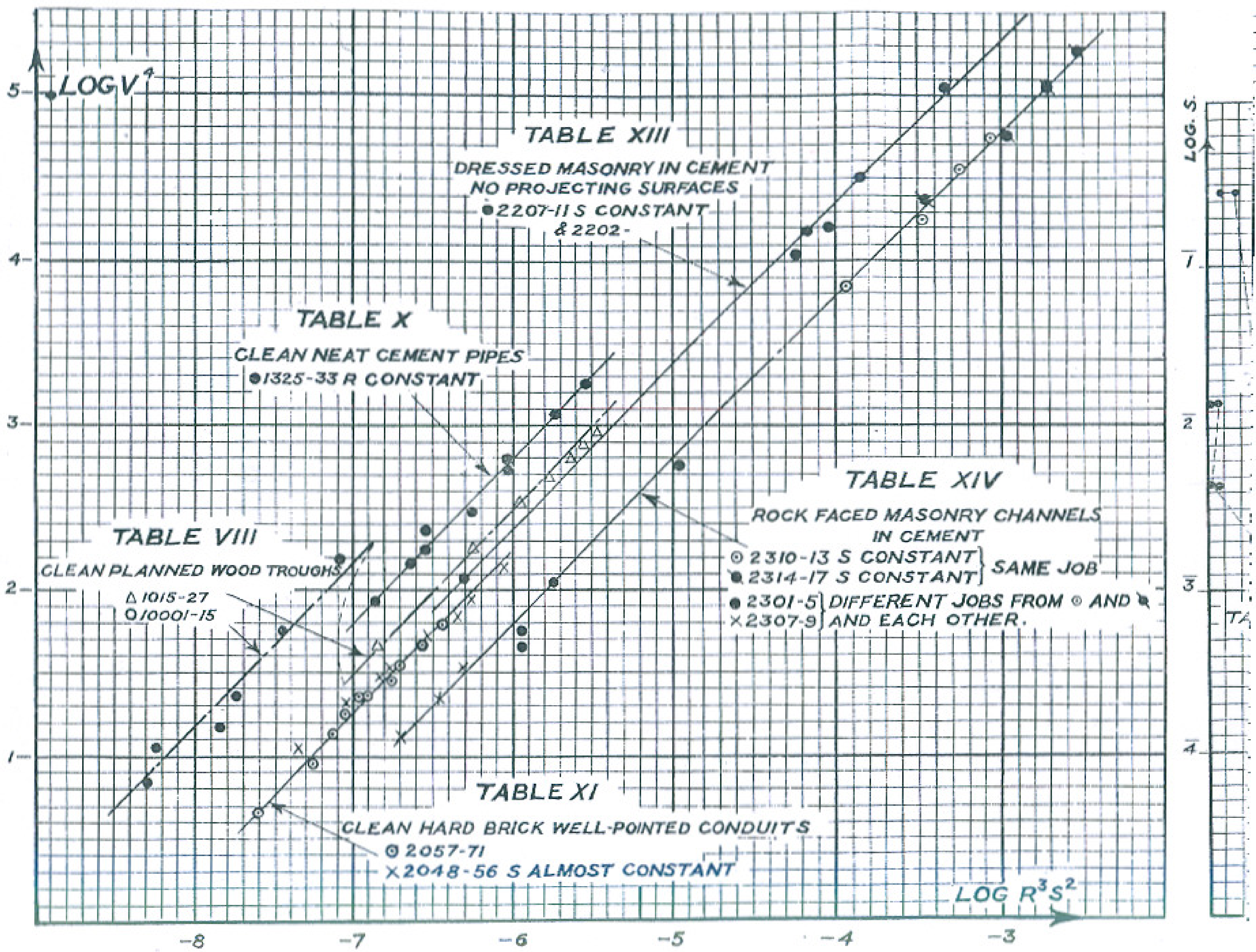
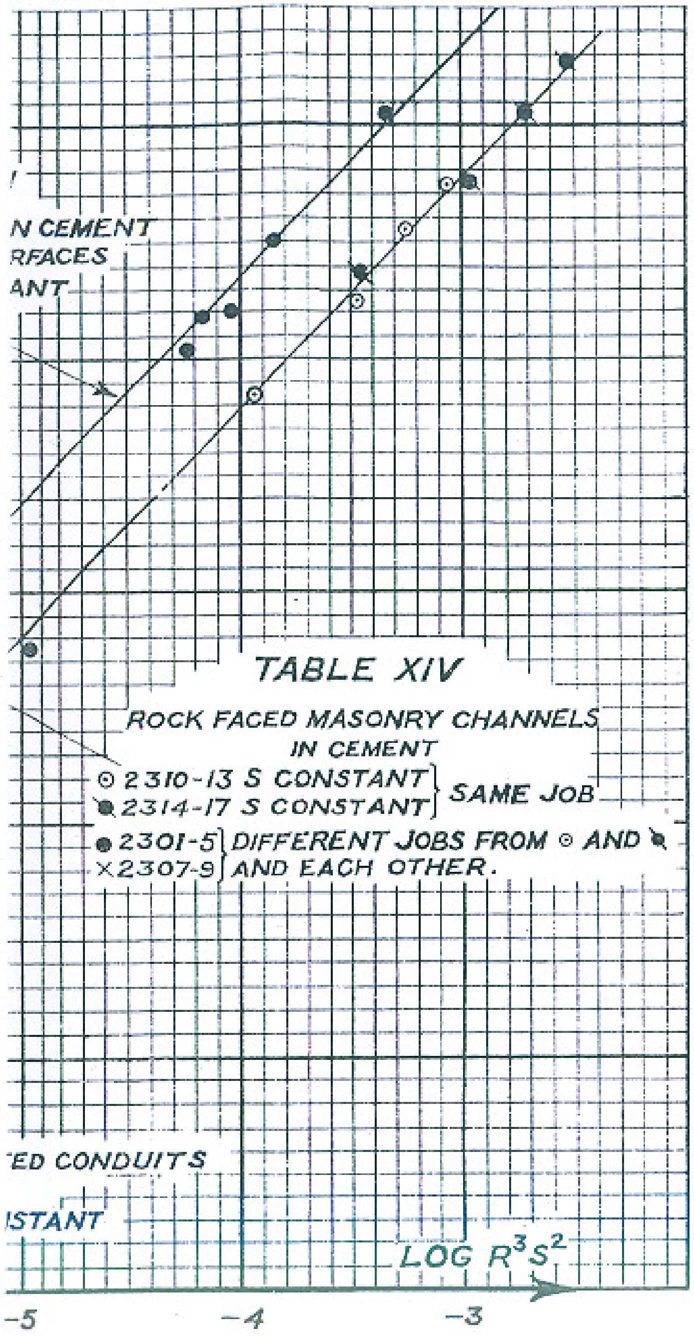
2. " Uniform Flow in Pipes". Hogan and Gibbs in " Concrete and Constructional Engineering" of November, 1936.



LOG V^4
5
4
3
2
1
TABLE XIII
DRESSED MASONRY IN CEMENT
NO PROJECTING SURFACES
● 2207-11 S CONSTANT
& 2202-
TABLE X
CLEAN NEAT CEMENT PIPES
● 1325-33 R CONSTANT
TABLE VIII
CLEAN PLANNED WOOD TROUGHS
△ 1015-27
○ 1001-15
TABLE XIV
ROCK FACED MASONRY CHANNELS
IN CEMENT
⊙ 2310-13 S CONSTANT } SAME JOB
● 2314-17 S CONSTANT }
● 2301-5 } DIFFERENT JOBS FROM ⊙ AND
× 2307-9 } AND EACH OTHER.
TABLE XI
CLEAN HARD BRICK WELL-POINTED CONDUITS
⊙ 2057-71
× 2048-56 S ALMOST CONSTANT
LOG R^3S^2
−8
−7
−6
−5
−4
−3
LOG. S.

FIG. I


N CEMENT
RFACES
ANT
TABLE XIV
ROCK FACED MASONRY CHANNELS IN CEMENT
⊙ 2310-13 S CONSTANT
● 2314-17 S CONSTANT
SAME JOB
● 2301-5
× 2307-9
DIFFERENT JOBS FROM ⊙ AND ●
AND EACH OTHER.
ED CONDUITS
ISTANT
LOG R^3S^2
-5
-4
-3

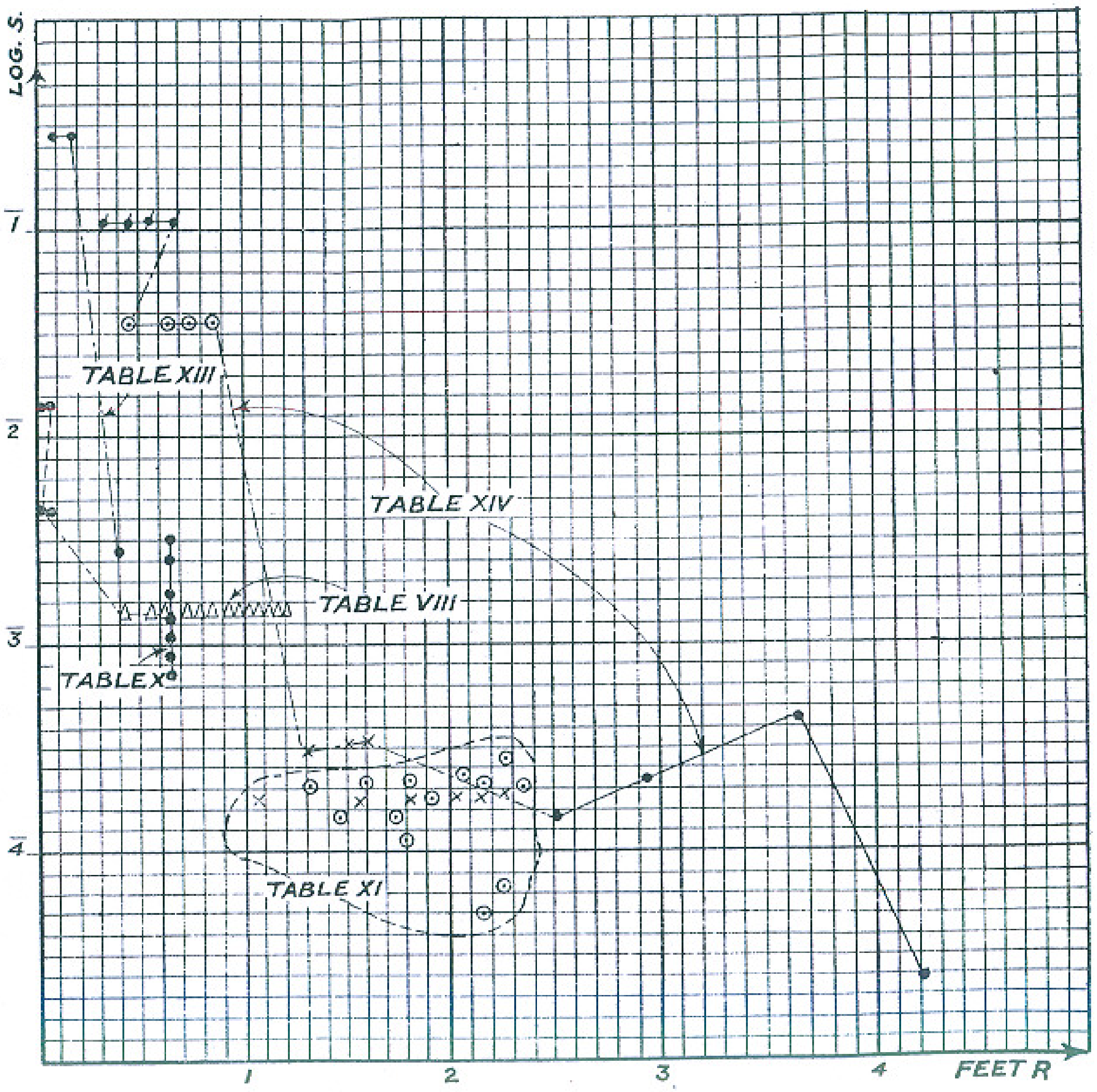

LOG. S.
$\bar{1}$
$\bar{2}$
$\bar{3}$
$\bar{4}$
TABLE XIII
TABLE XIV
TABLE VIII
TABLE X
TABLE XI
1
2
3
4
FEET R

FIG. I (a)

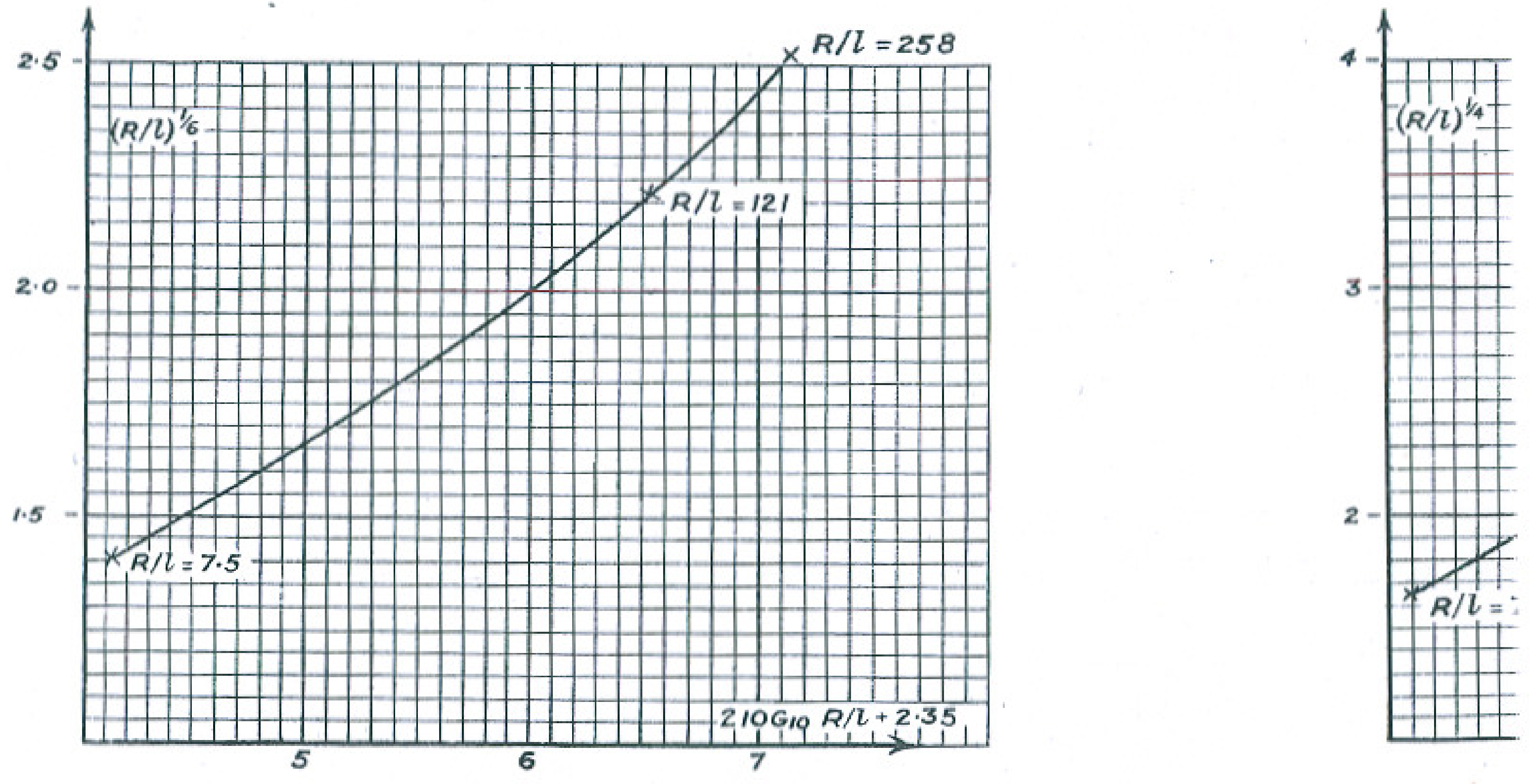
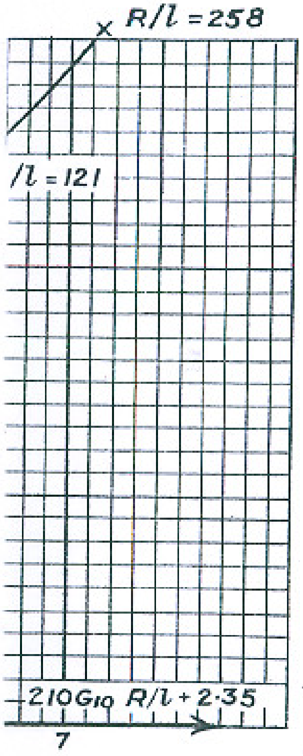


FIG. 2


R/l = 258
/l = 121
2 lOG10 R/l + 2·35
7

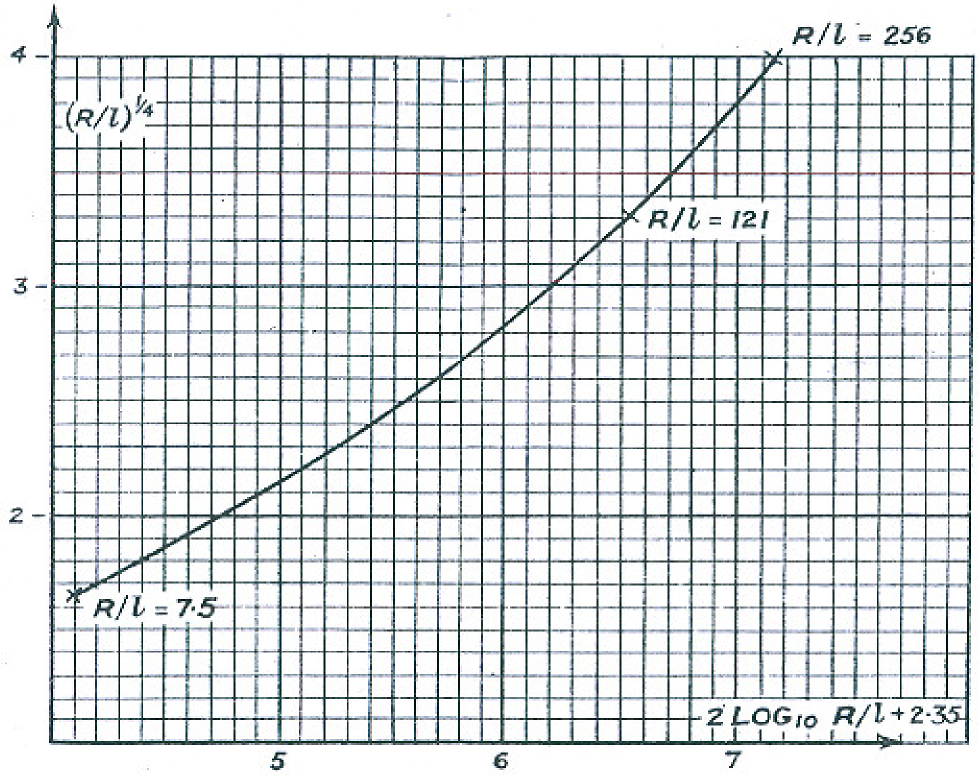
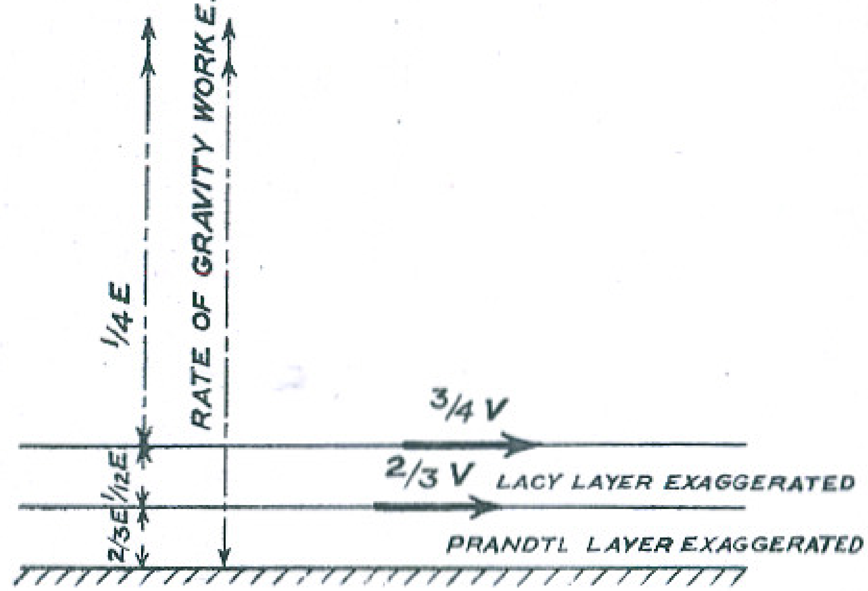

R/l = 256
(R/l)^1/4
R/l = 121
R/l = 7·5
2 LOG10 R/l + 2·35
4
3
2
5
6
7

FIG. 3

FIG. 4

PAPER NO. 212



PUNJAB ENGINEERING CONGRESS
1938

DISCUSSION.

The Author in introducing his Paper, explained that, although his results had been arrived at by applying somewhat advanced knowledge and methods, it was possible to show that existing accepted flow formulae pointed to the Universal Flow Formula V=abs. const.$(R/x)^{\frac{1}{4}}\sqrt{gRS}$, where x is the protuberance height for a " rough boundary", laminar film thickness for a " smooth boundary", and " equivalent protuberance" for an " incoherent boundary". He explained, in terms of a diagram of velocity distribution, the modern ideas on Relative Roughness, Laminar Film Thickness, and the physical difference between Roughness and Smoothness. Using Prandtl's relation that δ/R was of the order of (Reynold's Number)$^{-\frac{1}{2}}$ the Universal Formula reduced to V=abs. const.$(\nu/VR)^{\frac{1}{8}}\sqrt{gRS}$, which is the Formula of Blasius for smooth boundary using protuberance height for x, a formula almost the same as Manning's resulted. Using Equivalent Protuberance, the Lacey Flow Formula resulted.

He hoped that this simple demonstration of the truth of his results would appeal to those who had not the time to acquire the specialized knowledge requisite for a detailed understanding of the Paper.

Mr. **Thompson** remarked that in presenting Papers to the Congress the Authors of such Papers were evidently prompted to give to the other members of the Congress the result of their studies in the direction personal to the Authors.

In order that the other members derived some benefit from the communication, the matter should be presented to them in a form which was easily read and assimilated. Unless that property be satisfied the value of the contribution might well be questioned.

The present Paper, besides giving us the names of various prominent people dealing with the mathematical survey of the subject of the *Flow of Water* and listing various formulae, did not attempt to supply the want which such a Paper should satisfy.

In criticizing this method of presentation of the Paper it was necessary to anticipate the probable obvious reply that the members of the Congress should be so well informed as to follow with ease what was written in the Paper. To this the Speaker had to say that in his opinion the members of the Congress, especially the young members, were thoroughly well equipped to deal with or to read modern scientific literature, but no one could be expected to specialize and we did look to the specialist in various branches to make easy for the rest of us what presumbaly they understood thoroughly.

The Speaker did not wish the Author to think that these remarks of his had anything personal in their applications. The Speaker had considerable appreciation of the Author's talents but he did make a plea for the better presentation of what he wished to present to the other members of the Congress who were not so very well informed on the mathematics of the *Flow of Water*.

This Paper was not the first one which had been defective in the manner of presentation nor would it be the last unless those responsible for editing the Papers exercised a stricter control over the manner of presentation of Papers to the Congress.

From a perusal of the Paper it was gathered that the point at issue was whether the formula (1) was preferable to the formula (2). The difference between the two formulae amounted to a difference of 1/12 in the value of the co-efficient of 'R'.

After reading through the whole of the Paper, members still did not know which of the two was the better formula.

Mr. **Halcro Johnston** said that the object of this Paper namely to remodel the present flow formulae so that they would be dimensionally correct and would comply with theoretical requirements was excellent.

Most empirical formulae contained constants of which the dimensions were often complicated with different numerical values in each system of units; a troublesome calculation had to be made in changing from one system to another, and many different numbers had to be memorized; these disadvantages were removed when such constants were replaced by numerical constants of no dimensions.

As to the Author's conclusions, the Speaker had been unable to form an opinion as he had assumed a knowledge of the latest literature on the subject, which he did not possess. The Paper, the Speaker thought would have been much more valuable if the Author had justified his arguments more fully and had given in tabular form, at the beginning, the definitions of the letters used in the formulæ.

The Author appeared to consider that indices should be integers or simple fractions like $\frac{1}{2}$, wherever possible; in equation (15), for instance, he selected 2 and $\frac{1}{2}$ as indices of V and R without apparently any experimental justification; there appeared to be no theoretical justification for this preference for whole numbers. That it was theoretically unsound could readily be shown if an empirical formula of the form: $y=kx^n$ was made to fit one consisting of two terms, such as $y=Ax+Bx^2$, having integer indices. Nor were whole-number indices necessary from the point of view of dimensions; the right hand side of equation (15′), for instance, might be written,

$$\text{abs. const. } P\rho(gR)^{1-n}(l/R)^{n/2}V^{2n}.$$

This would be dimensionally correct whatever the value of n and practically equal to (15′) if n were approximately equal to 1.

Viscosity, the Author said, could not enter into equation (15); this statement needed justification. If the co-efficient of viscosity were to be reduced while the gradient S remained constant the water would accelerate due to reduced resistance and with a perfect fluid there would be no resistance.

The boundary resistance appeared rather as the cumulative effect of viscous flow throughout the liquid; the roughness of the boundary accentuated this effect by causing cross currents which resulted in differences in velocity in a direction at right angles to that vector and the greater these differences the greater the viscous resistance.

Equation (15) dealt with a cylindrical volume of water of unit length filling the whole pipe; if a cylinder of smaller diameter had been considered as flowing through the surrounding cylindrical shell of water, an equation similar to (15) would have been obtained but in this case the resistance would clearly have appeared as due to viscosity. The case treated by the Author could be considered as a limit of this general case when the diameter of the cylinder appoached that of the pipe; the difference between the two cases, therefore, appeared to be principally one of degree.

For the same reason it seemed incorrect to differentiate between the rates of internal work and of boundary work as the Author had done in equation (20).

The Speaker said he would try to support those statements by working out the case from first principles, starting with the equation of motion of a viscous fluid. Let us take for simplicity, he said, the case of a horizontal pipe.

Let $d\tau$ be a small element of volume,

v_τ its velocity vector,

x^τ its position vector,

p the pressure at the point,

F e force vector per unit mass due to gravity.

f_r the acceleration vector of the element.

$\frac{\partial p}{\partial x^r}$ the pressure gradient vector,

v^r the average velocity vector, equal to $\frac{\iiint v^r d\tau}{\iiint d\tau}$,

$\frac{\partial p}{\partial x^r}$ the average value of $\frac{\partial p}{\partial x^r}$ throughout the cylinder, (the scalar value of its horizontal component being $\rho g S$.)

$\iiint d\tau = A$, the cross-sectional area of the pipe.

The equation of motion of unit volume of the element could be written

$$\frac{\partial p}{\partial x^r} - \mu g^{st} V_{r,st} = \rho(F_r - f_r) \qquad \dots\dots (A)$$

(see McConnell's Applications of the Absolute Differential Calculus, page 281, equation (35)).

This was a covariant equation. Taking its scalar product with V^r, we would get the rate-of-work equation:

$$\frac{\partial p}{\partial x^r} v^r - \mu g^{st} v_{r,st} v^r = \rho(F_r - f_r) v^r \qquad \dots\dots (B)$$

Multiply by $d\tau$ and integrate throughout a cylinder of unit length. The righthand side of (B) equalled zero due to the following equations:

$\rho F_r \iiint v^r d\tau = \rho F_r v^r A = O$, since F_r and v^r were at right angles.

$\rho \iiint f_r v^r d\tau = O$, or was negligible, since the total acceleration was nil.

Only two terms were left in the rate-of-work equation;

$$\iiint \frac{\partial p}{\partial x^r} v^r d\tau = \mu \iiint g^{st} v_{r,st} v^r d\tau \qquad \dots\dots (C)$$

The left term was the rate of work due to the pressure gradient and appeared to equal $\rho g S V A$ as in equation (19); the right term was the rate of work due to viscosity; this clearly depended on μ, the co-efficient of viscosity. Also since both v^r, the velocity, and $v_{r,st}$, the rate of change of velocity gradient, would be affected by the roughness of the pipe, this term must also include the Author's index of roughness.

If, on the other hand, the equation of motion, (A), was multiplied by $d\tau$ only and integrated throughout the cylinder, we obtained the covariant vector equation:

$$\frac{\partial p}{\partial x^r}A-\mu\int\int\int g^{st}v_{r,st}d\tau=\rho F_r A \ldots\ldots\ldots\ldots\ldots\ldots\ldots (D)$$

Since F_r was vertical it did not enter into the horizontal component which was,

$$\rho gSA=\mu x \text{ (horizontal component of } \int\int\int g^{st}v_{r,st}d\tau) \ldots\ldots\ldots (E)$$

This was the Author's equation (15) and it also clearly showed that the resistance was the result of viscosity. Here also the value of $v_{r,st}$ would be determined by the roughness of the pipe.

Dr. **N.K. Bose** also spoke but regretted he was unable to remember what he said. (The following is taken from the Editor's rough notes and may not be quite correct in detail). Dr. Bose described an interesting talk he had with Prof. Prandtl (the world-renowned engineer) many years ago, *i.e.*, before Messrs. Lacey and Blench had published their researches, and Prof. Prandtl had assured him that nothing less than an exhaustive programme of research, divorced from all existing hypotheses and data would allow the scientist to enter the unknown world of turbulence, in connection with the transportation of silt. Dr. Bose was still of the opinion that Mr. Blench's work would lead nowhere. He also mentioned that it would be pertinent to ask why Mr. Lacey's name had been chosen for such an important characteristic of turbulence as Mr. Blench's "Lacey Layer".

Mr. **R. K. Khanna** remarked that Mr. Blench, while retaining that time-honoured privilege of trying to bluff himself and others that he was presenting a theory of flow of water, had dispensed with the necessity of stating the 'why' and 'wherefore'.

In the Author's own words, the object of his Paper was to find, by using Hamilton's Principle, the Energy Relations behind authoritative flow formulae, and thereby to establish a new flow formula for rough conditions of rigid channels and pipes. According to the Author, a further meaning was given to certain quantities whose exact nature was not known. Further, to prevent the analysis seeming indirect and involved, it was advisable, the Author said, to show first how, by using clues from its development and results, existing information could be corrected and correlated.

After the first mention of Hamilton's Principle in the opening paragraph of the Paper, and later on a reference to it in passing, Mr. Hamilton was quite lost to everybody. Indeed, nothing was said in the Paper about the "Energy Theory of Turbulent Flow of Liquids" itself, except in the title of the Paper. After stating the object of the Paper which was indeed laudable, the Author announced to the dazzled and bewildered members his own formula for rough flow as being

$$V = \text{Constant } R^{\frac{3}{4}} S^{\frac{1}{2}}.$$

After this sudden and unexpected appearance of the Author's formula, without a word of explanation as to how it was derived, there was probably no direct mention of the new energy theory of the Author in the remaining eleven pages of the Paper. The conclusions of the Author were detailed on the last page of the Paper.

For the ordinary reader, there was nothing but turbulence and incoherence to be found in the Paper!

Mr. **G. R. Sawhney** said that this Paper would have brought forth still a more interesting discussion, had at least the living scientists whose works had been made use of by the Author, to suit his own conclusions, had a chance of reading this Paper and offering their criticisms. This not being the case, the Author was luckier than the proverbial doctor because, in this case not only the dead could not speak but even the living, who were directly concerned, could not do so on the platform.

He thought such Papers were getting too advanced, both in the use of integral calculus and manipulation of indices, and also introduction of unknown co-efficients; and the resulting empirical formulae, rather than suggesting more common-sense methods of translating Nature and its laws and thus helping an average engineer to benefit by reading them and practising on them.

He would suggest that the word "suitable" would be more fitting than 'adjustable' for qualifying 'R' in formula (9).

The Lacey formula had been accepted but it was in his opinion no nearer the truth than the conclusion arrived at by other engineers and the designs made quite successfully by following them.

CORRESPONDENCE

Mr. **A. R. Thomas** communicated that he thought that:—

(1). The two impressions formed on a study of this interesting Paper were first the wealth of ideas and originality of treatment and second by the freedom with which quite important assumptions were made and gaps left in the argument, with scarcely any explanation.

In consequence of the latter it was difficult to say anything which was not merely a series of questions regarding the derivation of expressions and the justification for certain assumptions.

2. It was suggested that conventional symbols be used and an explanatory key provided.

3. On page 74 the Author mentioned the existence of a boundary layer, and stated that its relative thickness δ/R was of the order of $\sqrt{1/N}$ where N was the Reynold's number. This evaluation of the thickness was made use of later in the Paper. The boundary layer to which the Author referred was a layer along the boundary wherein the flow was laminar and viscosity had appreciable influence. It was unfortunate that this layer was referred to by many writers as the "boundary layer" as this term was in accepted terminology applied to the layer, developing from the leading edge of a solid body, within which due to viscosity the boundary had appreciable influence. The flow in the layer might be laminar or turbulent. In the case of a channel the boundary layer would increase in thickness from the upstream end until it enveloped the whole of the fluid. The layer to which the Author referred could more appropriately be termed the " laminar layer", as it was a layer adjacent to a solid boundary within which the flow was laminar, the flow outside it being turbulent.

4. Now the thickness of a *boundary* layer was of the order of $\sqrt{1/N}$ (Author's bib. 1, page 66) where N the Reynolds Number$=VL/\nu$ not VR/ν, L being the distance from the leading edge. An expression for the thickness of a *laminar* layer in a smooth pipe was given (bib. 1, page 79) as $\delta/\text{radius}=68{\cdot}4/N^{7/8}$, where N this time was, Vr/ν, r being the pipe radius, based on the Blasius's equation for resistance in smooth pipes. On what grounds did the Author arrive at $\sqrt{1/N}$ for the laminar layer?

5. It was of interest to examine the difference in boundary conditions between smooth and rough rigid boundaries. In each case a shear stress was to be transmitted from the boundary to the main body of the fluid in turbulent flow. Where the boundary was smooth turbulence could exist adjacent to it. The stress was therefore transmitted to the turbulent region through a laminar layer, i.e., directly by its viscosity. Viscosity therefore entered the expression for total resistance, which was proportional to the velocity raised to a power less than 2. Where the boundary was rough the flow around the protuberances forming the roughness was turbulent and the shear stress was transmitted by the reaction of normal pressure on them, and the change of momentum was transmitted into the fluid by turbulent mixing. In this case viscosity did not enter the expression for resistance, which was proportional to the square of the velocity, and there was no laminar layer. The quantity $V_*=\sqrt{\tau/\rho}$ which had the dimensions of a velocity, was used for

the comparison of velocities—to reduce velocities to a common non-dimensional basis.

6. The Blasius equation quoted by the Author for smooth pipes (3) had been shown to hold in the case of pipes only below a Reynolds Number of 10^5. For higher values of N an expression derived by Nikuradse which may be put into the form

$$V = \frac{\text{abs. constant}}{\rho f} \sqrt{RS},$$

where ρ = density of the water and $f = 0{\cdot}0032 + 0{\cdot}221\ N^{0{\cdot}237}$, was a better fit to the data (Rouse: Mechanics of Fluid Turbulence, Proc. Am. Soc. C. E., 1936).

7. When the Author on page 75 stated that one of the "more obvious defects" of the Prandtl-Karman-Nikuradse theory was the existence of a cusp in the velocity distribution curve, did he refer to the 1/7th root distribution law which was derived from the Blasius resistance equation (bib. 1, page 70)? If so, it seemed rather severe on Prandtl to say that his theory of resistance was defective because he derived quite independently of this a certain velocity distribution law from an equation which the Author apparently accepted as correct, particularly when it was not intended to apply to the whole cross section, as the Author must have applied it in finding the cusp at the centre.

8. Figure 2, comparing the Author's co-efficient with Prandtl's for rough boundaries, was stated to show that there was practically no difference as a fit to data. It would be of value in judging this point if the Author would plot the two formulae against the data.

9. On page 77 it was stated that $f_1 \propto (m\sqrt{gd}.g/(V_L\rho')$. It would assist the reader if the derivation of this expression, stated on page 83 to be obtained by dimensional analysis from $f_1 \propto d^{1/2}$, were given in more detail. What was ρ'?

10. On the same page $V_L = (\nu g^{1/3})$ was presented as a new concept arising from the Lacey theory. In the discussion on the Author's Paper of 1937 the writer suggested that in the equations of channels flowing in noncoherent bed material g would appear only in combination with S or $(\sigma - \rho)$, where σ = density of bed material. From his remark on page 85 it would appear that the Author, if he agreed, would associate $(\sigma - \rho)$ with g in this expression.

11. On page 78 under the heading of "Hamilton's Principle", δ did not refer to the thickness of the boundary layer but had its usual meaning in Calculus, *i.e.*, "a small increment in."

12. It was not clear why the principle of least squares showed that the kinetic energy of the relative motion must in any case be a minimum, and the Author was only justified in assuming that the kinetic energy of the mean motion must be a minimum if he could show that the two kinetic energies were a minimum under the same conditions. The root-mean-square velocity might not in "in practical cases" differ appreciably from the mean velocity, but as the Author's object was to show why what happened in practical cases did happen, it was suggested that account should be taken of the difference. In small channels the difference might be appreciable.

13. "For Lacey Regime Channels, $dV/dR=O=dV/dS$.". Was not shape considered also, *viz.*, $dV/d(P/R)=O$?

14. When the condition $dV/dS=O$ was applied to equation (20) the second energy term was omitted. The justification for this was not clear in view of the presence of V^m in the term. If both sides of (20) were multiplied by V/S and differentiated with respect to S a value of dV/dS was obtained which was not independent of the second term.

15. The Author's remarks on rigid rough channels appeared to be confined to channels of circular cross section. With cross-sections of other shapes although (R/l) might be constant the velocity gradient and the effective (R/l) were not constant over the wetted perimeter.

16. It was not clear how equations (9′) and (26) were derived from (9), nor (28) and (11′) from (11).

The Author's conception of two layers at the boundary, termed by him the Prandtl and Lacey layers, was of much interest. The former was presumably similar to the laminar layer in a channel with smooth boundary. The limit of a laminar layer was the plane where sufficient turbulence existed to render the effect of viscosity negligible—an arbitrary limit. In the Author's conception what defined the limits of the Lacey layer and was the flow therein laminar or turbulent? If the bed material were wholly confined below the turbulent layer no silt would be carried into suspension, except perhaps a small quantity by saltation. The writer visualized the bed as comprising a series of waves of bed material, the flow being laminar on the upstream of each wave carrying material with it to the crest from where the fine particles were carried into the turbulent region above while the coarse particles settled into the trough. Particles were continuously settling from above on to the upstream slopes, with the result that the waves might move downstream or upstream according as the rate of settlement was less or greater than the rate of rising into suspension. Did the Author visualize the bed as a plane surface as would appear from Figure 4, and if so how did he explain the carrying of particles into suspension?

17. The essence of the Paper was the application of Hamilton's Principle as an underlying law governing the tendencies of channels and pipes in adjusting their dimensions. This appeared to be sound and opened a line of reasoning which should lead to considerable advances. If the Author would rewrite or augment his Paper, explaining intermediate steps and pointing out assumptions made, stating their justification, it should be one of great value, and it was hoped that he would continue his research in the subject.

The **Author** in replying to the discussions said that he appreciated Mr. Thompson's point of view but regretted it was impossible to simplify a specialized subject to such an extent that it would be immediately understood by a general audience. He had tried to make amends in his introduction, by giving a simple bibliography from which specialized knowledge could be obtained easily. The Paper itself contained only simple mathematics and ideas and a study of the Bibliography would remove most of the apparent difficulties with the possible exception of Hamilton's Principle, which the average engineer might find difficult to derive *ab initio*.

The difference between Manning's and the Author's formulae was greater than Mr. Thompson thought. If they agreed at $R=1{\cdot}0$ foot, they would disagree by about 20 per cent. at $R=9{\cdot}0$ feet, so Manning's formula would not be suitable for prediction from small scale experiments of the behaviour of even moderate-sized tunnels. The practical value of the new formula was, however, incidental; its real value lay in the fact that it was dynamically sound, and its derivation led to new physical ideas and the unification of what was previously disjointed. Since writing the Paper the Author had derived a velocity distribution consistent with the Flow Formula, using von Karman methods, and considered that he had now obtained a complete theory of turbulence of original physical interest.

He was most interested to see Mr. Halcro Johnston employing the beautiful notation of the Tensor Calculus, although not agreeing in full with his interpretation of equation (B). $\iiint F_r v^r d\tau$ was not zero, for F_r and v^r were not at right angles. The integral gave the rate of working by gravity per lb. The integral $\int\int\int \frac{\partial p}{\partial x^r} v^r d\tau$ gave the rate of working per lb. by the pressure-gradient. In an open channel the pressure-gradient was nil. In a horizontal pipe the rate of gravity work was nil, for then gravity was really at right angles to flow. The Author avoided using two terms by using S for non-dimensional pressure gradient, or for channel slope. The use of non-dimensional terms had greatly simplified hydraulics, just as the use of tensor notation had simplified physics in general. The rate of working of internal forces was given by $\mu\int\int\int g^{st} v_{r,st} . v^{r} . d\tau$. The trouble, the Author said, with this integral was that it could not be evaluated without knowing velocity distribution in turbulence, which was precisely what we should like to find but could not. The standard use of Green's Theorem would make matters no simpler, for it would give

$$\int\int\int g^{st} v_{,srt} v^{r} d\tau = \int\int v^{r} . v_{r,s} d\sigma - \int\int\int g^{st} v_{t}^{\ r} v_{r,s} d\tau.$$

It was just this inability to evaluate the viscous work integral that had driven modern investigators, such as von Karman, to approach the subject by analysing authoritative experimental data in the light of dimensional analysis, and sound general dynamics, in the hopes of obtaining functions with an obviously definite meaning (*e.g.*, Reynold's Number, or V_*, or Lacey's V^2/R).

These would allow systematization of previously disjointed data, and act as pointers to the nature of the mechanism that gave rise to them. The tremendous advance given by these new methods was most apparent in von Karman's work, set out simply and appreciatively in Bakhmeteff's "The Mechanics of Turbulent Flow" (Princeton University Press, 1936). The Author had used the new method of attack as follows: (*i*) replaced the viscous work integral by the two parts, one for the Laminar Film, and another for the Inner Fluid; (*ii*) used the minimal condition behind Hamilton's Principle, as applicable to the nature of uniform turbulent flow in a pipe or channel. The particularization inherent in (*ii*) was still not very detailed; but it was sufficient to obtain the ratio between the two parts of the viscous work integral to derive a boundary velocity law, indicate the general resistance law for all cases, give a definite meaning to Lacey's V^2/R, and give valuable hints on other matters. The results were of considerable scientific and practical value; but the full expression of the viscous work integral in such a way as to give the rate of working at any time for any part of the fluid would never be possible by any method, for detailed turbulence was erratic, although the bulk results were systematic.

The establishment of a ratio between the parts of the viscous work integral explained why viscosity could be dispensed with. Physically the reason was that a *definite proportion* of the gravity rate of work was transferred into vortex form. This proportion, given sufficient knowledge, should be expressible in terms of the viscosity that damped out the energy; but it was practicable to use an expression of the cause of the vorticity which arose from impact against protuberances, and was therefore expressible without viscosity at all. An analogue was that of an aeroplane propeller. The rate of working could be expressed in terms of the engine driving it, or the rate of dissipation of energy in the air stream produced.

The Author regretted he had not given full reasons for insistence on simple indices, and Mr. Johnston was quite correct in stating that a complicated index might result from the attempt to replace a simple algebraic function by a power of one variable. It was, however, established by all authoritative experiments that "rough boundary" condition was a definite physical condition in which $S \propto V^2$. The range of S was too great for this to be disputed. Figure I (*a*) would support this, for a logarithmic scale had to be used to make the diagram manageable. Transition data, from smooth to rough boundary conditions, both of which were stable states with simple indices, did actually have complicated indices because their "law" was a combination of the simple indicial laws of the extreme stable states.

Replying to Dr. Bose, the Author condemned his negative attitude, and referred him to para. 16 of Bakhemteff's "Mechanics of Turbulent Flow", to impress on him that von Karman's theory was not accepted as final even by its Author. He said that a writer of Bakhemteff's ability could discuss the defects and, at the same time, appreciate the tremendous advance obtainable from a new theory based on sound physical ideas, giving such an excellent approximation to truth, and systematization where none existed before. He assured Dr. Bose that he had not misquoted the Lacey theory in any way.

Messrs. Khanna and Sawhney were thanked for their remarks.

The Author, from informal discussions with certain officers, felt that he had caused some confusion by changing nomenclature. He had used "Mixing Velocity" instead of "Shear Stress Velocity", and "Boundary Layer" instead of "Laminar Film". The order of thickness of the Laminar Film in this Paper was derived by the identical method used by Prandtl to obtain the order of thickness of the Boundary Layer for a plate.

Reply to Correspondence by Mr. A. R. Thomas. The Author thanked Mr. Thomas for his appreciation, and for the very detailed and useful points raised. They were best replied to in paragraphs, numbered to suit the queries.

1. Would be attended to in further detailed publications.

2. The symbols followed Prandtl's reasonably closely.

3. For "boundary layer" read "laminar film." The terms appeared to have acquired specialized meanings, and the confusion caused by mixing them was regretted. Prandtl's argument in para. 39 of Bib. 1 applied exactly to the Laminar Film, although it was worked out with special reference to the Boundary Layer.

4. The method of Bib. 1, page 79 was highly artificial. δ was given a conventional definition, the seventh root law was now out-of-date (having been superseded by von Karman's logarithmic one and being variable with N), and the assumption $u_{max}=1{\cdot}235_u$ was also inaccurate. Bib. 1 gave several *formal* definitions of δ and each would lead to a different formula. The method of para. 39 was much the best as it consisted of omitting second order terms in the equation of Navier-Stokes and was free from any conventionality.

5. Agreed.

6. It was unfortunate that some investigators had tried to "improve" the equation of Blasius to cover physical conditions where it was inapplicable. When N had increased sufficiently to reduce δ to about protuberance height the boundary conditions were no longer smooth. The flow equation could not then be simply expressed, as it was a combination of the equation for purely smooth and purely rough boundaries, a different combination for every value of N. Increase of N would eventually result in purely rough boundary conditions and the flow formula would then be definite once more.

In short, equation (7') was for rough boundary, equation (8) was for smooth; the former gave $S \propto V^2$, and the latter gave $S \propto V^{1{\cdot}75}$; the formulae were exact for the conditions they represented, and no formula devised to smooth the one into the other, *viz.*, the transition, could possibly be exact for either. Rouse's formula was based on a misunderstanding of the physics of the case.

7. The reference was to von Karman's logarithmic curve, not mentioned in Bib. 1, but in Bakhmeteff's "Mechanics of Turbulent Flow", referred to hereafter as Bib 3. The 1/7th root law was now obsolete.

8. The data were not available at the time of writing. If Fig. 3 be compared with Bib. 3, Fig. 60 where the data was plotted it would be seen that $R/l=7{\cdot}5$ and 126 were for transition and smooth conditions. What was left of Fig. 3 after excluding these was pactically straight, and cut the axis of $R/l=0$ at a distance of $0{\cdot}6$ from the origin. The zero error was adjustable by changing the method of measuring l. Thus if mean height was half total height and we measured mean height instead of total height the zero error was removed, as $0{\cdot}6$ was almost exactly 2. log 2. The labour of plotting data was therefore unnecessary. The formulae were indistinguishable over the range of Nikuradse's experiments.

9. May be done directly, without m and ρ', by standard methods, or from the consideration that the dimensions of $\sqrt{gd}$ were those of

velocity, which were cancelled by those of V_L, leaving g which was an acceleration like f. Then multiply by m/ρ' which was non-dimensional. Physical considerations showed that m must occur, and ρ' was required to balance it. (See also eqn. (8) of Lacey's replies to I.C.E. Paper 4893). ρ' was a density, but must be associated with both silt and water in a way the Author could not define exactly at present.

10. This expressed the Author's opinion exactly, and he owed it largely to points raised by Mr. Thomas on a previous paper. σ and ρ were now, however, relative densities.

11. Noted; but the symbol was common to hydraulics and calculus.

12. The difference mentioned was slight, but led to considerable mathematical difficulties. The Author would be interested in data of small channels showing a large difference. Mr. Halcro Johnston had shown the Author that the principle of least squares was not required. In fact the K. E. of the fluctuating motion being, on an average, a fixed fraction of that of the mean motion, for any one case, the vanishing of the differential coefficient for the whole implied the vanishing for the parts.

13. Yes, the condition $dV/d\,(P/R)=O$ must have an application; but not to the Energy Equation, as P cancelled out. An equation about the silt-water film was wanted which, the Author thought, treated by the condition mentioned would lead to $P=2{\cdot}67Q^{\frac{1}{2}}$.

14. The differential coefficient V^m contained dV/dS as a factor, *i.e.*, zero as a factor. Multiplication by V/S complicated the work but must give the same answer.

15. R was an omnibus term which ought to get over the difficulty except for abnormal shapes. This view found support in Bib. 1, page 43.

16. Taking (28) as a simple example and writing

$$V^3 \propto g^a \nu^b R^2 S$$

dimensionally this gave

$$L^3T^{-3}=(LT^{-2})^a(L^2T^{-1})^b L^2,$$

$$\text{i.e.,}\quad a+2b=1,$$

$$2a+b=3,$$

Whence $a=\frac{5}{3}$ and $b=-\frac{1}{3}$,

So that $V^3 \propto (g^{5/3}\nu^{1/3})\,R^2S$

$$\propto (g^2/V_L)R^2S \ldots\ldots\ldots q.\ e.\ d.$$

The term C was introduced for physical reasons.

The other formulae were equally simple.

17. The Author in no way disagreed with Mr. Thomas's conception; but would prefer not to dogmatize on what was still partly speculative. The Author had only established that the nature of the boundary could be represented by a linear dimension in the universal flow formula, just as for rough rigid and smooth rigid boundaries. Even in a rough boundary the linear dimension must contain a shape factor and a spacing factor; but, in practice, we were not interested in these matters. We merely recorded that boundaries of certain general natures gave certain values of C in $V=CR^{3/4}S^{1/2}$(C being=abs. constant $/l^{1/4}$). Theoretically, the nature of l, or t, was of great interest; and experiments on l were being made by various workers. Mr. Thomas's interest in the nature of t might be expected to lead to an equation from which $P=2{\cdot}67Q^{1/2}$ could be derived by using $dV/dP=0$. The Author was in full general agreement with Mr. Thomas's ideas on silt-movement and emphasized that t, like l, or d, or f, was an omnibus term defining boundary nature, but not yet correlated with the more detailed mechanism behind it.

18. A Paper developed from the viewpoint of the non-irrigation engineer had already been prepared, and accepted by the I.C.E. It had succeeded in working out a velocity distribution law in terms of a vortex model, and the result was in complete accordance with the form of the Universal Flow Formulae.